A family of hyperchaotic multi-scroll attractors in $\mathbf{R}^n$

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ABSTRACT

In this work, we present a mechanism how to yield a family of hyperchaotic multi-scroll systems in $\mathbf{R}^n$ based on *unstable dissipative systems*. This class of systems is constructed by a switching control law changing the equilibrium point of an unstable dissipative system. For each equilibrium point presented in the system a scroll emerges. The switching control law that governs the position of the equilibrium point varies according to the number of scrolls displayed in the attractor. Thus, if two systems display different numbers of scrolls, they have different switching control laws. This paper also presents a generalized theory that explains different approaches such as hysteresis and step functions from a unified viewpoint, extending the concept of chaos in $\mathbf{R}^3$ to hyperchaotic multi-scroll systems in $\mathbf{R}^n, n \geqslant 4$. An illustrative example of synchronizing a communication system is given based on the developed theory.

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1. Introduction

Chaos has been an extremely well-studied area in the last decades: some recent approaches are based on suppression or induction of chaotic behavior, others focus on synchronization [1,2] and the generation and analysis of time series with the purposes of implementing them for modulation schemes or encrypting them in communication systems [3].

So far, chaotic behavior may be generated in two kinds of nonlinear systems. The first one includes continuous systems with nonlinearities given by multiplication of their states, and the second one presents a combination of piecewise-linear (PWL) systems.

The characterization [4], implementation, and design of new switched systems of chaotic behavior [5], especially possessing multiple scrolls or wings [6,7], has been of great interest for the scientific community. The methods implemented to generate multi-scroll systems in the literature may be catalogued in two: (i) systems presenting more equilibrium points than wings or scrolls, (ii) systems presenting the same number of equilibrium points and wings or scrolls.

The present study focuses on generation of a family of systems with multi-scroll attractors based only on PWL systems. This family presents three remarkable properties as follows.

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1.1. An hyperchaotic multi-scroll system.

A characterization of dynamical behavior can be achieved by means of the Lyapunov exponents. Using their values, one can determine the average exponential rates at which nearby orbits diverge or converge. Their signs define a qualitative picture of dynamics that systems may exhibit, ranging from fixed points via limit cycles and tori to more complex chaotic and hyperchaotic attractors. Whereas chaos can arise in discrete-time systems with only a single state (which must be positive), at least three state variables are required to generate chaos in continuous-time systems [8]. Such systems are characterized by one positive exponent in the Lyapunov spectrum. However, in order to obtain hyperchaos, the system must be characterized by the presence of at least two positive Lyapunov exponents. The reason is that the trajectory has to be nonperiodic and bounded within a finite region and cannot intersect itself.

Since the hyperchaotic Rössler attractor was reported in [9], many studies have been focused on generation of hyperchaotic systems [10,11], and their synchronization problem [12]. The methodology used to generate this behavior in benchmark systems, such as Lorenz and Rössler ones, is via nonlinearities of the system. The class of systems considered in this paper possesses, besides being hyperchaotic, the following property.

1.2. For each equilibrium point introduced into the system, a scroll emerges in the attractor.

It is known that the generation of multi-scroll behavior in PWL systems is based on the location of their equilibrium points. Their commutation surfaces or thresholds bound the scrolls and give a specific direction to the flow. Various papers on this topic have presented different theories developed to explain how to generate multi-scroll chaotic attractors. This paper develops an approach to generate chaotic attractors based on unstable dissipative systems.

Since the work reported by Suykens in [13] about n-double scrolls in the Chua's system, there have been different approaches to yield multi-scroll attractors. These approaches vary from modifying the nonlinear part in the Chua's system [14,15,13,16–18], to using nonsmooth nonlinear functions, such as hysteresis [19,20], saturation [21,22], threshold, and step functions [23–29]. Recently, fractional-order systems have also been used to generate multi-scroll attractors [30–32].

There have been also reports on generation of hyperchaotic multi-scroll behavior in [33–35], where the number of equilibrium points is greater than the number of scrolls. In addition to presenting hyperchaotic attractor with an equal number of scrolls and equilibrium points, the considered class of systems possesses the following third property.

1.3. The family of hyperchaotic multi-scroll systems can be extended to $\mathbf{R}^n, n \geqslant 4$.

Yalçin et al. [25] reported that 1D, 2D and 3D-grid of scrolls may be introduced locating them around the equilibrium points using a step function. Lu et al. in [19,20] presented an approach based on hysteresis that enables the creation of 1D n-scrolls, 2D $n \times m$-grid scrolls, and 3D $n \times m \times l$-grid scrolls.

In this work, continuing [29], we develop a generalized theory capable of explaining different approaches as hysteresis and step functions and extending the concept to hyperchaotic multi-scroll systems to $\mathbf{R}^n, 4 \leqslant n \in \mathbb{Z}$.

This family of systems is composed of *unstable dissipative systems* (UDS) [29,36]. Since such a system is unable to provide a stable flow by itself (due to unstable saddle points among its equilibria), a switching control law (SCL) is designed to change from one UDS to another and, by this mechanism, generate a PWL system with a multi-scroll attractor. Each scroll results from an equilibrium point and an unstable "one-spiral" trajectory it produces.

This paper is organized as follows: In Section 2, we introduce a theory explaining the generation of multi-scroll behavior via UDS. Some examples are given. Section 3 presents a family of hyperchaotic systems in $\mathbf{R}^n, n \geqslant 4$. In Section 4, we compare some of the previously known approaches to the designed UDS-based method. Section 5 presents a communication system based on the synchronization of UDS systems. Section 6 concludes this study.

2. Generation of multi-scroll attractors by UDS

We consider the class of linear system given by

$$\dot{\boldsymbol{\chi}} = \mathbf{A}\boldsymbol{\chi} + \mathbf{B} \tag{1}$$

where $\boldsymbol{\chi} = [x_1, x_2, \ldots, x_n]^T \in \mathbf{R}^n$ is the state variable, $\mathbf{B} = [B_1, B_2, \ldots, B_n]^T \in \mathbf{R}^n$ stands for a real vector, $\mathbf{A} = [\alpha_{ij}] \in \mathbf{R}^{n \times n}$ with $i, j = 1, 2, \ldots, n$ denotes a linear operator (matrix). The equilibrium point is located at $\chi^* = -\mathbf{A}^{-1}\mathbf{B}$. The system dynamic is given by the matrix $\mathbf{A}$, which has a stable manifold E^s and an unstable one E^u. On the basis of the previous discussion, it is possible to define two types of UDS as follows:

Definition 2.1. We say that system (1) is an UDS of Type I if $\sum_{i=1}^n \lambda_i < 0$, where $\lambda_i, i = 1, \ldots, n$, are eigenvalues of $\mathbf{A}$, and at least one λ_i is a negative real eigenvalue, and at least two λ_i are complex conjugate eigenvalues with positive real part $Re\{\lambda_i\} > 0$.

Definition 2.2. We say that system (1) is an UDS of Type II if $\sum_{i=1}^n \lambda_i < 0$, where $\lambda_i, i = 1, \ldots, n$, are eigenvalues of $\mathbf{A}$, at least one λ_i is a positive real eigenvalue, and at least two λ_i are complex conjugate eigenvalues with negative real part $Re\{\lambda_i\} < 0$.

Note that systems with these characteristics present two kinds of saddle equilibrium points. An UDS of Type I presents a stable fast eigendirection, due to the real negative eigenvalue, and an unstable spiral slow eigendirection, due to the positive real part of the complex conjugate eigenvalues. An UDS of Type II presents an unstable fast eigendirection, due to the real positive eigenvalue, and a stable spiral slow eigendirection, due to the negative real part of the complex conjugate eigenvalues. Fig. 1 illustrates the behavior of each of the two types of UDS. In this paper, we focus our study only on UDS of Type I.

The next proposition follows from the general theory of linear systems.

Proposition 2.1. *Let the system* (1) *be an UDS of Type I with ordered eigenvalues set* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$. *Then, the system has a stable manifold* $E^s = span\{\lambda_1, \lambda_2, \ldots, \lambda_j\} \subset \mathbf{R}^n$ *and an unstable one* $E^u = span\{\lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_n\} \subset \mathbf{R}^n, 1 \leqslant j \leqslant n$, *and the following statements are true:*

(a) *Any initial condition* $\boldsymbol{\chi}_0 \in \mathbf{R}^n/E^s$ *leads to an unstable orbit that goes to infinity.*
(b) *Any initial condition* $\boldsymbol{\chi}_0 \in E^s$ *leads to a stable orbit that converges at* $\boldsymbol{\chi}^*$ *and the system does not generate oscillations.*
(c) *The basin of attraction is* $E^s \subset \mathbf{R}^n$.

Now, we consider a switching system based on the linear system (1) given by

$$\dot{\chi} = \mathbf{A}\chi + \mathbf{B}(\chi),$$

$$\mathbf{B}(\chi) = \begin{cases} B_1, & \text{if } \chi \in \mathcal{D}_1; \\ B_2, & \text{if } \chi \in \mathcal{D}_2; \\ \vdots & \vdots \\ B_k, & \text{if } \chi \in \mathcal{D}_k, \end{cases} \tag{2}$$

where $\mathcal{D}_i$ are such that $\mathbf{R}^n = \cup_{i=1}^k \mathcal{D}_i$. Thus, the equilibria of the system (2) are $\chi_i^* = -\mathbf{A}^{-1}B_i$, with $i = 1, \ldots, k$. The objective is to define vectors B_i assuring the stability of a class of dynamical systems in $\mathbf{R}^n$ with oscillations within an attractor. In other words, for any initial condition $\boldsymbol{\chi}_0 \in \mathbf{R}^n$, the orbit $\phi(\boldsymbol{\chi}_0)$ of the system (2) is trapped in an hyperchaotic attractor $\mathfrak{A}$ upon defining at least two vectors B_1 and B_2. This class of systems can display various multi-scroll attractors as a result of a combination of several unstable "one-spiral" trajectories, i.e., we are interested in a vector field that can yield multi-scroll attractors by switching vectors $B_i, i = 1, \ldots, k$ and $k \geqslant 2$. We assume that each domain $\mathcal{D}_i \subset \mathbf{R}^n$ contains the equilibrium $\chi_i^* = -\mathbf{A}^{-1}B_i$. Following the preceding discussion, we can define a multi-scroll chaotic system based on UDS of Type I.

Definition 2.3. Consider a system given by (2) in $\mathbf{R}^n$ and equilibrium points $\chi_i^*, i = 1, \ldots, k$ and $k \geqslant 2$. We say that system (2) is a multi-scroll system with the minimum of equilibrium points, if each χ_i^* observes oscillations around it and for any initial condition $\chi_0 \in R^n$ the orbit $\phi(\chi_0)$ generates an attractor $\mathfrak{A} \subset \mathbf{R}^n$.

To illustrate our approach, we consider a particular case of the linear ordinary differential equation (ODE) written as $\dddot{x} + \alpha_{33}\ddot{x} + \alpha_{32}\dot{x} + \alpha_{31}x + \beta_3 = 0$, representing the state equation (2), where the matrix $\mathbf{A}$ and the vector $\mathbf{B}$ are defined as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ \beta_3 \end{pmatrix}. \tag{3}$$

The characteristic polynomial of the matrix $\mathbf{A}$ given by (3) takes the form $\lambda^3 + \alpha_{33}\lambda^2 + \alpha_{32}\lambda + \alpha_{31}$.

Defining the coefficients as

$$\alpha_{31} = 1.5, \quad \alpha_{32} = 1, \quad \alpha_{33} = 1. \tag{4}$$

results in the set of eigenvalue $\Lambda = \{-1.20, 0.10 \pm 1.11i\}$, which satisfy Definition 2.1 and ensure that the system is an UDS of Type I. The component β_3 of the vector B is governed by a switching control law (SCL), which can be designed depending on the number of scrolls to be introduced. A SCL for 2 scrolls is represented as:
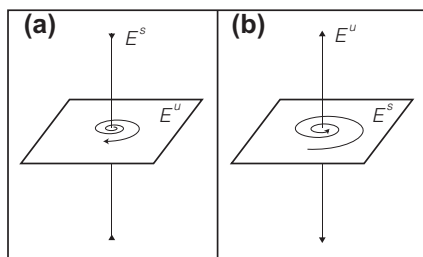

Fig. 1. Dynamics of UDS: (a) Type I, (b) Type II.

$$\beta_3 = \begin{cases} 1.8, & \text{if } x_1 \geqslant 0.3; \\ -0.9 & \text{otherwise}. \end{cases} \tag{5}$$

The equilibrium points of the system (2) using the matrix **A** and vector **B** defined in (3) and the SCL (5) are $\chi_1^* = (1.2, 0, 0)^T$ and $\chi_2^* = (-0.6, 0, 0)^T$. Note that the number of equilibrium points coincides with the number of scrolls in the attractor.

Fig. 2(a) depicts the projection of the 2-scroll attractor onto the (x_1, x_2) plane generated by the signal u from (5) under Eqs. (2) and (3) and initial conditions $\chi_0 = (1, 0, 1)^T$.

Now, if we change the control signal given by the SCL, then it is possible to generate an attractor with a desired number of scrolls. The next example presents a 4-scroll. For this purpose, β_3 can be assigned as follows:

$$\beta_3 = \begin{cases} 1.8, & \text{if } x_1 \geqslant 0.9; \\ 0.9, & \text{if } 0.3 \leqslant x_1 < 0.9; \\ 0, & \text{if } -0.3 < x_1 < 0.3; \\ -0.9, & \text{if } x_1 \leqslant -0.3. \end{cases} \tag{6}$$

The parameter β_3 given by the SCL (6) introduces other two equilibrium points located at $\chi_3^* = -\chi_2^*$ and $\chi_4^* = (0, 0, 0)^T$. Fig. 2(b) shows the projection of the 4-scroll attractor given by the SCL (6). Note that displacement of the scrolls is shown along the x_1 axis. Introducing more equilibrium points along different axes and designing a specific SCL would result in any desirable number of scrolls inside a n-dimensional grid. For example, for an unstable system (1) and a SCL given by (5), a typical orbit is shown in Fig. 3.

3. Generation of a family of hyperchaotic multi-scroll attractors in $\mathbf{R}^n$

Here, we present a family of hyperchaotic multi-scroll attractors, based on the Type I UDS theory described before. The class of linear systems is given by (1) in $\mathbf{R}^n$, with $n \geqslant 4$. First, considering $n = 4$, we define the state variable as $\chi = [x_1, x_2, x_3, x_4]^T \in \mathbf{R}^4$, adding a negative feedback in the equation for x_4, so the matrix **A** and vector **B** are defined as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ \beta_3 \\ \beta_4 \end{pmatrix}. \tag{7}$$

The characteristic polynomial takes the form $(\lambda + 1)(\lambda^3 + \alpha_{33}\lambda^2 + \alpha_{32}\lambda + \alpha_{31})$. With the parameter values assigned in (4), the set of eigenvalues becomes $\Lambda = \{-1.00, -1.20, 0.10 \pm 1.11i\}$, where a negative real eigenvalue has been added. Thus Definition 2.1 is satisfied and the system given by (2) and (7) is an UDS of Type I.

For the sake of simplicity, we consider β_3 equal to β_4. Then the control signal may be described by any of the SCLs given before ((5) or (6)). Both SCLs make the system (2) and (7) behaving as a multi-scroll system with the minimum number of equilibria for a PWL system.

Using the algorithm and based on the definition introduced by Wolf *et. al.* [37], the Lyapunov exponents for the system (2) and (7) are two maximum positive exponents $(0.10223, 0.10119)$, which demonstrate that the system is hyperchaotic. These exponents remain the same regardless of the SCL applied to the system.
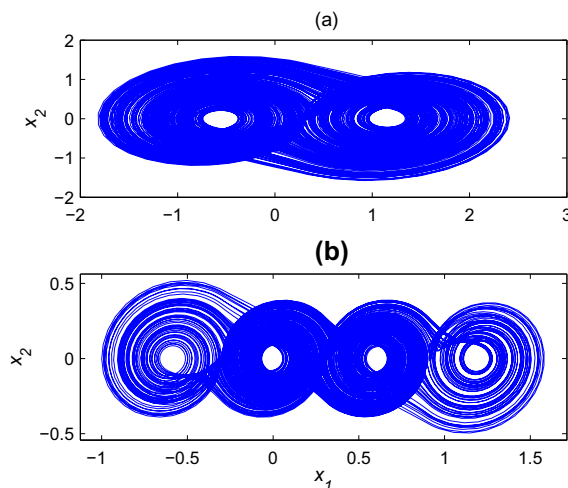

Fig. 2. The projections onto the (x_1, x_2) plane of the attractors generated by different control signals: (a) (5), (b) (6), with the coefficients values defined in (4).
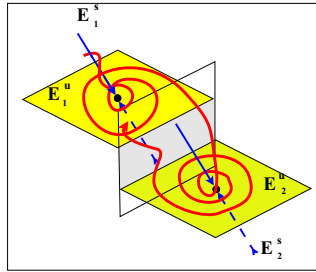
Fig. 3. Typical orbit behavior for a two equilibria UDS of Type I.

Fig. 4 shows the projections of the hyperchaotic attractor onto the $(x_1, x_2), (x_1, x_3)$ and (x_1, x_4) planes, which is generated by the Eqs. (2) and (7) with initial conditions $\chi_0 = (1, 1, 0, 0)^T$ and SCLs corresponding to:

(a) 3-scroll attractor

$$\beta_3 = \begin{cases} 0.9, & \text{if } x_1 \geqslant 0.3; \\ 0 & \text{if } -0.3 < x_1 < 0.3; \\ -0.9, & \text{if } x_1 \leqslant -0.3. \end{cases} \tag{8}$$

(b) 5-scroll attractor

$$\beta_3 = \begin{cases} 1.8, & \text{if } x_1 \geqslant 0.9; \\ 0.9, & \text{if } 0.3 \leqslant x_1 < 0.9; \\ 0, & \text{if } -0.3 < x_1 < 0.3; \\ -0.9, & \text{if } -0.9 < x_1 \leqslant -0.3; \\ -1.8, & \text{if } x_1 \leqslant -0.9. \end{cases} \tag{9}$$

The equilibrium points of system (1), (7) are $\chi^* = [\beta_3/\alpha_{31}, 0, 0, \beta_3]^T$. Since the equilibrium points are controlled by (8) and (9), they are located in the plane (x_1, x_4), as observed in Fig. 4(c) and (f). In addition, choosing an appropriate SCL makes it possible to generate any desirable number of scrolls in the system.

So far, we have worked with the parameters defined in (4). Varying the parameter α_{31} in (7), we determine for which values Definition 2.1 is satisfied. Fig. 5 corresponds to the range $-2 \leqslant \alpha_{31} \leqslant 2.5$. For values of α_{31} less than 0, the system pos-
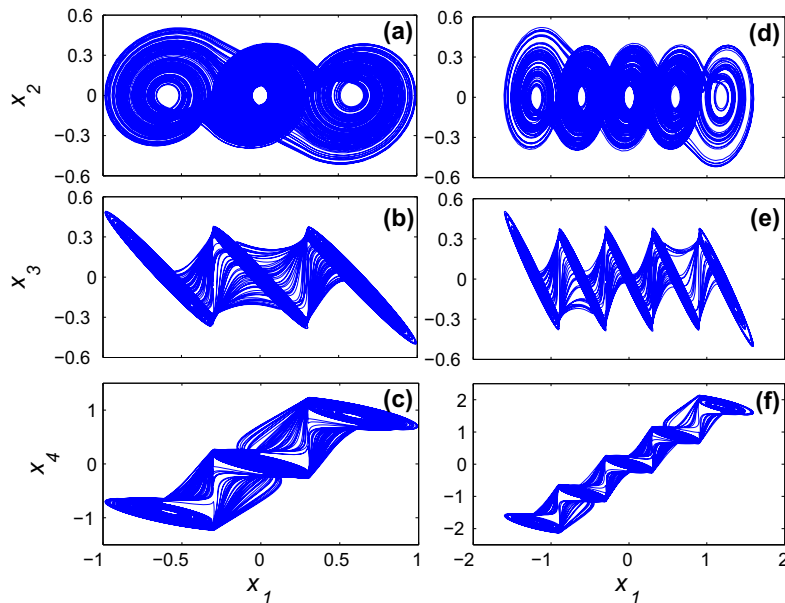

Fig. 4. Hyperchaotic multi-scroll attractors in $\mathbf{R}^4$ generated by the system (2) and (7), using the values defined in (4) with different control signals. Applying the SCL (8), the projections of the 3-scroll attractor are shown onto: (a) (x_1, x_2) plane, (b) (x_1, x_3) plane, (c) (x_1, x_4) plane. Applying the SCL (9), projections of the 5-scroll attractor are shown onto: (d) (x_1, x_2) plane, (e) (x_1, x_3) plane, (f) (x_1, x_4) plane.

sesses one positive real eigenvalue, one negative real eigenvalue, and two complex conjugate eigenvalues with a negative real part. Therefore, the system is an UDS of Type II of Definition 2.2. In the area between 0 and approximately 1, the real parts of the eigenvalues are negative. This means that the system is stable and any equilibrium point for this region is a sink. Multistability holds in this area for different initial conditions. Finally, for values of α_{31} greater than approximately 1, the system possesses two real eigenvalues, one positive and one negative, and two complex conjugate eigenvalues with positive real part. Therefore the system is an UDS of Type I for $\alpha_{31} > 1$.

We can consider any system of the form (2) and (7) with parameters $\alpha_{31} > 1, \alpha_{32} = 1, \alpha_{33} = 1$ as a hyperchaotic multi-scroll UDS of Type I.

The design procedure can be extended to greater dimensions. For instance, considering $n = 5$, the state variable is $\chi = [x_1, x_2, x_3, x_4, x_5]^T \in \mathbf{R}^5$, and the matrix $\mathbf{A}$ and the vector $\mathbf{B}$ are defined as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix}.$$

(10)

Here, setting $\beta_3 = \beta_4 = \beta_5$ can be given by the SCLs (5), (6), (8), or (9). This system, obtained from (7) by introducing a negative feedback into the equation for x_5, possesses the characteristic polynomial $(\lambda + 1)^2(\lambda^3 + \alpha_{33}\lambda^2 + \alpha_{32}\lambda + \alpha_{31})$. Two more negative real eigenvalues were introduced; however, the form of the polynomial is similar to those in $\mathbf{R}^3$ and $\mathbf{R}^4$. Using the same approach, this result can be generalized to any dimension $n \geqslant 4$ with the following system parameters:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ -\alpha_{31} & -1 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 & -1 \end{pmatrix},$$

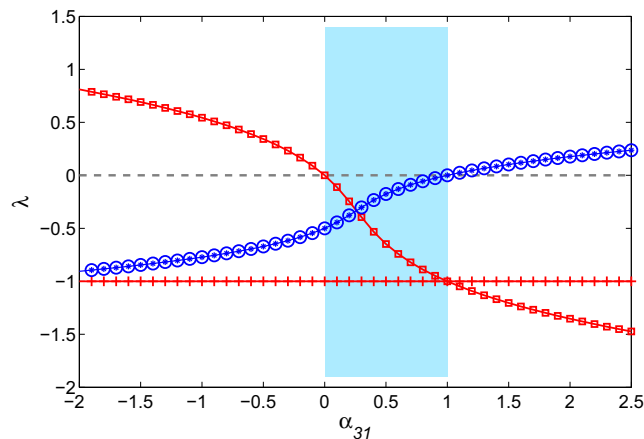$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_n \end{pmatrix}.$$

(11)

Fig. 5. Eigenvalues of the system given by (2) and (7) as functions of the parameter a_{31}. The real eigenvalues are marked with squares and crosses, and the real part of the complex conjugate eigenvalues is marked with circles and dots. For $a_{31} >\sim 1$, the system is an UDS of Type I.

Definition 3.1. Consider a system given by (2) and (11) in $\mathbf{R}^n$ with $n \geqslant 4$, which possesses a characteristic polynomial of the form $(\lambda + 1)^{(n-3)}(\lambda^3 + \lambda^2 + \lambda + \alpha_{31})$. We say that a system belongs to a family of hyperchaotic multi-scroll unstable dissipative systems, if $\alpha_{31} >$ and $\beta_3, \beta_4, \ldots, \beta_n$ define such a SCL that each equilibrium point χ_i^* observes oscillations around it, and the flow $\phi(\chi_0)$ generates an attractor $\mathcal{A} \subset \mathbf{R}^n$.

Regardless of the system dimension, the two maximum Lyapunov exponent remains positive, thus determining that any system in $\mathbf{R}^n$ belongs to the family. The method to obtain the proposed family of hyperchaotic multi-scroll in $\mathbf{R}^n$ can be summarized as follows:

- First, select an $\mathbf{R}^3$ module as described in (3).
- Select parameters $\alpha_{31}, \alpha_{32}, \alpha_{33}$ to assign eigenvalues according to Definition 2.1.
- For each new degree of freedom introduced into the system, select the system matrices **A** and **B** as described in (11).
- Design the SCL $\mathbf{B}(\chi)$ in (2), taking into account the distance between the equilibrium points of the introduced subsystems. Each equilibrium point results in a scroll.

We conducted an electronic circuit implementation of a multi-scroll attractor in $\mathbf{R}^4$. The description and electronic scheme are available from the authors upon request.

4. Unified theory of multiscroll attractors in R^n

Using the UDS-based approach, it is possible to develop a unified theory that includes different methods proposed for generation of hyperchaotic multi-scroll attractors. Here, we analyze such methods as hysteresis [19] and step function [25]. Other generation methods can be found in [19–21,24,25,27].

4.1. Step function

Yalçin [25] used a step function to generate 1D, 2D, and 3D-grid scroll attractors. The system (3), (4) with the parameters implemented in [25] ($\alpha_{31} = 0.8, \alpha_{32} = \alpha_{33} = 1$) results in the eigenvalue set $\Lambda = \{-0.89, 0.04 \pm 0.94i\}$ satisfying Definition 2.1. Here, the equilibrium points are located along the x_1 axis depending on the number of scrolls.

To include this method in the framework of the study of the UDS of Type I in $\mathbf{R}^4$, we transform the step function to incorporate it into the system (7). The step function takes the following form:

$$\beta_3(x_1) = \sum_{i=1}^{M_x} g_{\frac{(-6i+3)}{10}}(x_1) + \sum_{i=1}^{N_x} g_{\frac{(6i-3)}{10}}(x_1),\tag{12}$$

where

$$g_\theta(x_1) = \begin{cases} 0.9, & \text{if } x_1 \geqslant 0 \ \theta > 0 \\ 0, & \text{if } x_1 < 0 \ \theta > 0 \\ 0, & \text{if } x_1 \geqslant 0 \ \theta < 0 \\ -0.9, & \text{if } x_1 < 0 \ \theta < 0. \end{cases}\tag{13}$$

Here, $M_x, N_x \in \mathbf{R}$ correspond to the numbers of equilibrium points added to the system from the left or from the right of the origin, respectively, along the x_1 axis. The function $g_\theta(x_1)$ represents the step size, which is related directly to the switching surface of the system. Fig. 6 shows the projections of the attractors generated by the step function onto the (x_1, x_2) plane and the (x_1, x_4) plane. Since the SCL parameter β_3 affects only the states x_1 and x_4, the equilibrium points are located along the x_1 axis (see Fig. 6(a)) and on the plane (x_1, x_4) (Fig. 6(b)). The initial condition for the system is $\chi_0 = (1, 1, 0, 0)^T$.

4.2. Hysteresis

Lü et al. [19], designed a hysteresis series based on a step function to obtain multi-scroll chaotic attractors in 1-D n-scroll, 2-D $n \times m$-grid scroll, and 3-D $n \times m \times l$-grid scroll. The hysteresis series is given by Eqs. (1) and (3) in [19].

This approach can also be considered under the UDS definition of Type I. Analyzing the matrix **A** in (3) with the parameters given in [19] ($\alpha_{31} = 0.8, \alpha_{32} = 0.72, \alpha_{33} = 0.6$), results in the eigenvalue set $\Lambda = \{-0.85, 0.12 \pm 0.95i\}$, satisfying Definition 2.1. The equilibrium points for the 1-D n-scroll are located along the x-axis.

We use the parameters (4) in the matrix (7) to produce hysteresis based on a step function in $\mathbf{R}^4$. The corresponding SCL is governed by the parameters β_3 and $\beta_4 : \beta_3$ takes the form:
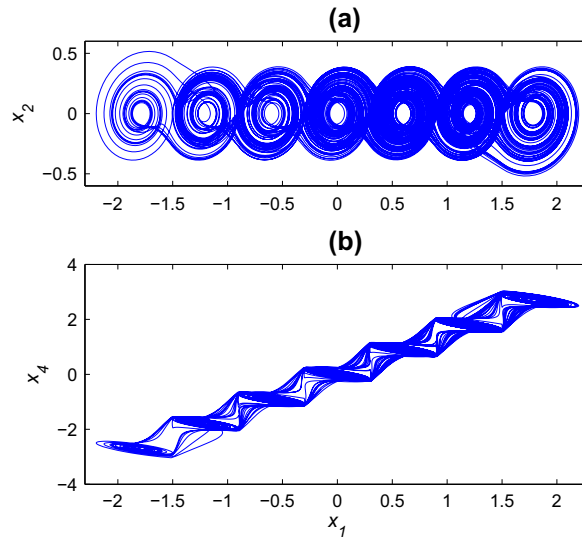
(a)

(b)

Fig. 6. The projection of the attractors generated by the step function (13) with $M_x = N_x = 3$ onto (a) (x_1, x_2) plane, (b) (x_1, x_4) plane.

$$
\beta_3 = \begin{cases}
-3.6, & \text{if } x_1 < -2.1 + \sigma\delta, \\
-2.7, & \text{if } -2.1 + \sigma\delta < x_1 < -1.5 + \sigma\delta, \\
-1.8, & \text{if } -1.5 + \sigma\delta < x_1 < -0.9 + \sigma\delta, \\
-0.9, & \text{if } -0.9 + \sigma\delta < x_1 < -0.3 + \sigma\delta, \\
0, & \text{if } -0.3 + \sigma\delta < x_1 < 0.3 + \sigma\delta, \\
0.9, & \text{if } 0.3 + \sigma\delta < x_1 < 0.9 + \sigma\delta, \\
1.8, & \text{if } 0.9 + \sigma\delta < x_1 < 1.5 + \sigma\delta, \\
2.7, & \text{if } 1.5 + \sigma\delta < x_1 < 2.1 + \sigma\delta, \\
3.6, & \text{if } x_1 > 2.1 + \sigma\delta
\end{cases}
\tag{14}
$$

and $\beta_4 = 0$. Here, $\sigma = sgn(\frac{d}{dt}x_1)$ denotes the direction on the x_1 axis where the system moves to, and $\delta \in \mathbf{R}$ is a constant value that defines hysteresis properties. A variation in δ increases or decreases the hysteresis region of the system, as depicted in Fig. 7. The resulting multi-scroll attractor may also change according to variations of this parameter, this numerical simulations were implemented with the subroutine ode45 from MATLAB. Fig. 8(a) presents a bifurcation diagram of the state x_1 vs δ. The initial condition for the system is $\chi_0 = (1, 0, 1, 1)^T$. It can be observed that for $\delta \approx 0$ the SCL makes the system behaving as a 9-scroll attractor. In the interval $0.1 \leqslant \delta < 0.8$, the system also exhibits multi-scroll, but the switching surfaces generating the scrolls are displaced by δ. There are intervals near some values of δ, where the attractor appears to collapse to only one scroll, for example, $\delta = 0.05$ and $\delta = 0.235$. These intervals correspond areas of multistability. Fig. 8(b) presents a zoom for $0 \leqslant \delta < 0.5$, where the multi-scroll is displayed in more detail. Fig. 9 shows the projection onto the (x_1, x_2) plane for $\delta = 0.15$. It can be concluded that the hysteresis series based on a step function acts exactly as the SCL in UDS of Type I described previously in (2) for $\delta = 0$.

5. Communication system

In this section, we implement a communication system based on chaotic modulation, as proposed in [38]. The idea is to synchronize two identical UDS systems in $\mathbf{R}^4$ given by Eqs. (1) and (7), using the complete replacement design from Pecora and Carroll [39], as depicted in Fig. 10. Although there are several approaches that can be used to synchronize a pair of chaotic systems [1,2,40], here we employ the one described by Pecora and Carroll, since this method allows only one signal to be transmitted reducing the risk of eavesdropping. The master system is given by:

$$
\dot{\chi}_m = \mathbf{A}\chi_m + \mathbf{B}_m\mathbf{u}_m,
\tag{15}
$$

where $\chi_m = [x_{1m}, x_{2m}, x_{3m}, x_{4m}]$ is the state variable and the linear operator $\mathbf{A}$ takes the same form as in (7). Here, we consider that $\mathbf{B}_m = [b_{i,j}] \in \mathbf{R}^{4\times2}, i = 1, 2, \ldots, 4, j = 1, 2$, is a real matrix and $\mathbf{u}_m = [u_{j,1}] \in \mathbf{R}^{2\times1}$ represents both SCL and external input as follows:

$$
\mathbf{B}_m = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{u}_m = \begin{pmatrix} \beta_3 \\ \xi \end{pmatrix},
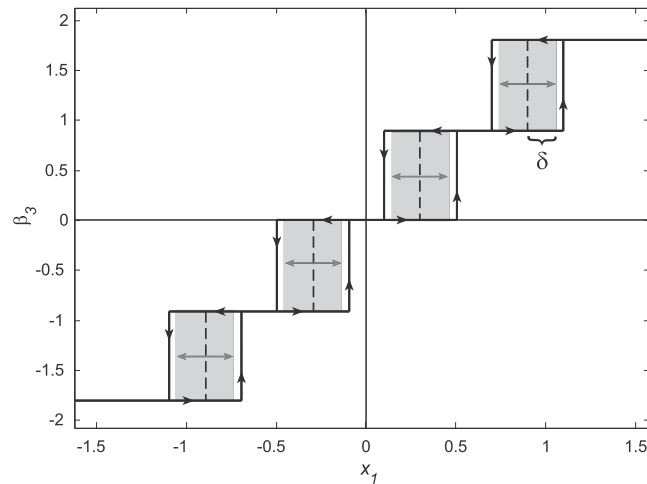\tag{16}
$$

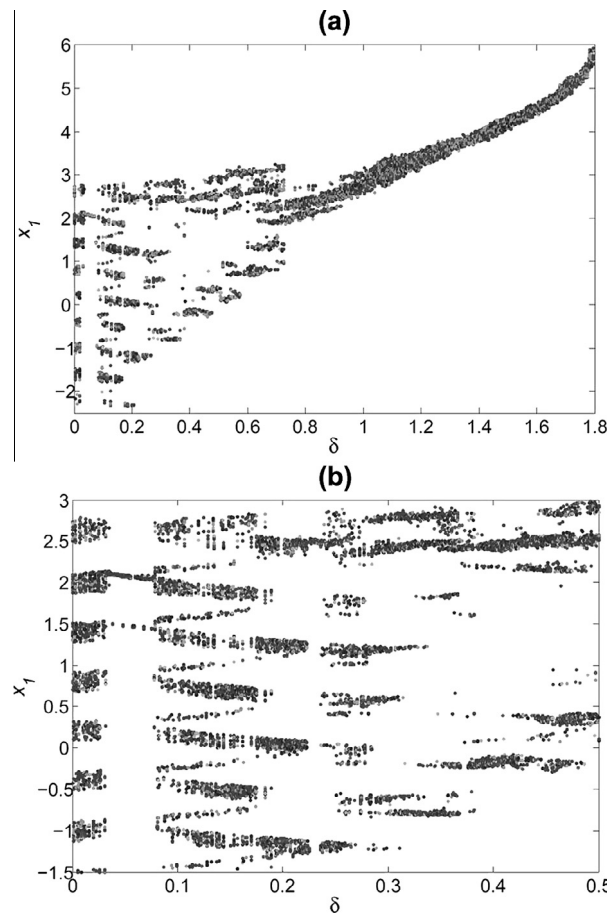**Fig. 7.** Diagram of the hysteresis series (14).

Fig. 8. Bifurcation of the attractors generated by the variation of δ along the x_1 axis, in the hysteresis series (14): (a) $0 > \delta > 1.8$, (b) $0 > \delta > 0.5$.

where ξ is analog information signal which affects the master system. We take x_{1m} from the master system and use it to re-place the state of the slave system initialized with different initial conditions. The synchronized subsystem is given by $\dot{\boldsymbol{\chi}}_r = \mathbf{A}\boldsymbol{\chi}_r + \mathbf{B}_r\mathbf{u}_r$, where $\chi_r = [x_{1m}, x_{2r}, x_{3r}, x_{4r}]$, and $\mathbf{A}, \mathbf{B}_r$, and $\mathbf{u}_r$ are the same as in (7).

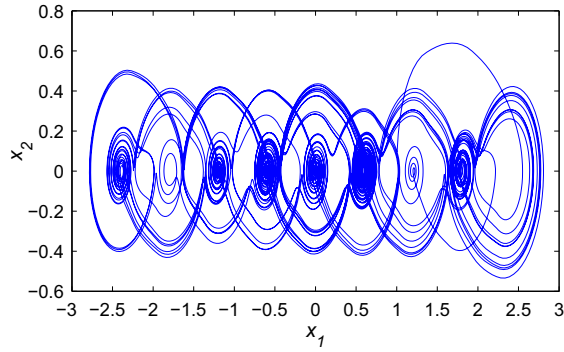As a result, the master and slave systems take the following form:

Fig. 9. The projection of the attractors generated by the hysteresis series (14) for $\delta = 0.15$ onto the (x_1, x_2) plane.
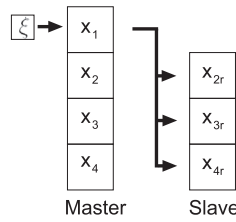

Fig. 10. Complete replacement synchronization scheme with chaotic modulation of the information input signal ξ.
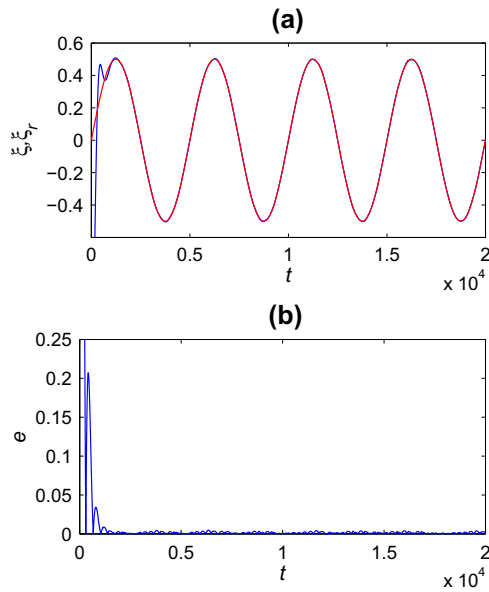

Fig. 11. (a) Modulated ξ (red) and reconstructed ξ_r (blue) signals. (b) Error between ξ and ξ_r. (For interpretation of the references to colour in this figure caption, the reader is referred to the web version of this article.)

$$
\begin{aligned}
\dot{\chi}_{1m} &= x_{2m} + \xi, \\
\dot{\chi}_{2m} &= x_{3m}, \\
\dot{\chi}_{3m} &= -a_{31}x_{1m} - a_{32}x_{2m} - a_{33}x_{3m} + \beta_3, \\
\dot{\chi}_{4m} &= -x_{2m} - x_{4m} + \beta_3, \\
\chi_{1r} &= \chi_{1m}, \\
\dot{\chi}_{2r} &= x_{3r}, \\
\dot{\chi}_{3r} &= -a_{31}x_{1m} - a_{32}x_{2r} - a_{33}x_{3r} + \beta_3, \\
\dot{\chi}_{4r} &= -x_{2r} - x_{4r} + \beta_3,
\end{aligned}
\tag{17}
$$

where β_3 is defined in Eq. (5). Here, the state x_{1m} of the master system is the carrier signal, which is transmitted to the slave one. Given that χ_m and χ_r are identical, the entire hyperchaotic systems would be synchronized if $\lim_{t\to\infty}\|\chi_m - \chi_r\| \to 0$. Accordingly, the reconstruction of the information signal could be made as $\xi_r = \frac{d}{dt}x_{1m} - x_{2r}$.

The original information signal is given by $\xi = 0.5sin(40\pi t)$ and is depicted in Fig. 11(a), along with the reconstructed signal ξ_r. The initial conditions are assigned as $\chi_{m0} = (1,0,0,0)^T$ and $\chi_{r0} = (1,1,1,0)^T$. Fig. 11(b) shows the error $e = \|\xi - \xi_r\|$ between the original and reconstructed signals. It can be observed that the error diminishes very rapidly, thus proving the feasibility of the developed approach.

6. Conclusions

This work presents a method based on UDS of Type I to generate a family of hyperchaotic attractors that possess multiscrolls in $\mathbf{R}^n$. Various techniques to yield multiscrolls, such as step function and hysteresis, can be covered by this approach. The multi-scroll systems generated here require a prior characterization of the equilibria for each of the subsystems contained within. Therefore, one can design a SCL that generates the flow between domains of unstable subsystems belonging to the PWL system.

Taking this in consideration, we recall one of the main features pertinent to the UDS of Type I. For each equilibrium point added to the system, a scroll emerges. This presents a significant difference from systems consisting of UDS of both types, which observe two scrolls for each series made of two systems of Type I and one of Type II. Using the developed approach, a communication scheme has been implemented based on synchronizing two UDS systems.

Acknowledgments

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