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“LOS HIPERESPACIOS T-CERRADOS Y T_n-CERRADOS”

TESIS POR ARTÍCULOS ESPECIALIZADOS

QUE PARA OBTENER EL GRADO DE:

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P R E S E N T A:

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Contents

Resumen

Un continuo es un espacio métrico, compacto, conexo y no vacío. La teoría de Hiperespacios de Continuos es una linea de investigación en Topología que tiene sus inicios con los trabajos de F. Hausdorff y L. Vietoris. Los hiperespacios son ciertas colecciones de subconjuntos de un continuo X con alguna característica particular.

A finales de los años 30, Burton Jones motivado por el estudio de la aposindesis definió la función $\mathcal{T} : P(X) \to P(X)$ en un continuo X de la siguiente forma, dado A un subconjunto de X

$$\mathcal{T}(A) = \{x \in X : \text{para cada subcontinuo } W \text{ tal que } x \in Int(W), \text{ entonces } W \cap A \neq \emptyset\}.$$

esta función fue definida con el objetivo de estudiar propiedades de los continuos, Jones demostró varias propiedades de esta función y caracterizó algunas clases de continuos usando esta función, desde entonces esta función ha sido ampliamente estudiada y es conocida como la función $\mathcal{T}$ de Jones.

El concepto de un conjunto $\mathcal{T}$-cerrado fue definido y estudiado por D. P. Bellamy et al. en [1], donde la $\mathcal{T}$ denota la función $\mathcal{T}$ de Jones. En este trabajo definimos el hiperespacio $C_{\mathcal{T}}(X)$ para un continuo X, como la colección de todos los subcontinuos $\mathcal{T}$-cerrados de X, analizamos el hiperespacio $C_{\mathcal{T}}(X)$ de algunos continuos, en particular del producto de dos continuos, también estudiamos la idempotencia de la función $\mathcal{T}$ en continuos, con lo cual obtuvimos varios resultados, los cuales se incluyeron en el articulo "Some aspects related to the Jones' set function $\mathcal{T}$", el cual fue publicado en la revista Topology and its Applications.

Siguiendo con el estudio del hiperespacio $C_{\mathcal{T}}(X)$ construimos un continuo X tal que $C_{\mathcal{T}}(X)$ es homeomorfo al conjunto de Cantor. Analizamos la conexidad, la compacidad, y densidad del hiperespacio $C_{\mathcal{T}}(X)$ y además dimos una caracterización de los continuo de la clase(W) estos y otros resulta-

dos fueron incluidos en el articulo "The hypersapce of $\mathcal{T}$-closed subcontinua", el cual fue enviado a la revista Topology and its Applications para su publicación.

Protocolo

Resumen

Analizaremos el concepto de triodo fuerte el cual fue introducido en el trabajo que realicé durante mis estudios de maestría, este análisis nos va a ayudar a caracterizar el interior de los triodos bajo funciones suprayectivas en el espacio de representación. También trabajaremos con los hiperespacios de los subcontinuos T-cerrados y T_n-cerrados los cuales definimos por primera vez en este proyecto.

INTRODUCCIÓN

La Teoría de Continuos es una rama de la Topología que se encarga de estudiar las propiedades de los espacios métricos, compactos, conexos y no vacíos; a un espacio con dichas características se le llama continuo, de igual forma a un subconjunto A de un espacio X se llamará un subcontinuo de X sí a su vez A es un continuo. La definición original de continuo, fue dada por G. Cantor en 1883 que dice que un continuo es un subconjunto perfecto (es decir cerrado y denso en sí mismo) de un espacio euclidiano. Esta área de las matemáticas nace aproximadamente en la década 1910-1920, con los trabajos de F. Hausdorff y L. Vietoris. Una subdivisión de la Teoría de Continuos es la Teoría de Hiperespacios de Continuos. Un hiperespacio de un continuo X, es una colección de subconjuntos de X que satisfacen propiedades específicas. La Teoría de hiperespacios es una herramienta auxiliar que nos permite estudiar propiedades del continuo a través del cual se definen, y en sí misma, es un mundo nuevo en el cual se pueden desarrollar teorías independientes.

El presente proyecto de doctorado esta dividido en dos partes, el objetivo de la primera parte es estudiar el concepto de triodo fuerte y los objetivos que pretendemos en la segunda parte es hacer un estudio minucioso de los continuos que tienen una cantidad finita o numerable de continuos T_n-cerrados.

ANTECEDENTES

El espacio de Representación fue estudiado por primera vez en el Primer Taller Polaco-Mexicano en Teoría de Continuos en el que participaron los Doctores José G. Anaya, Enrique Castañeda, Félix Capulín, Wlodzimierz, J. Charatonik y Fernando Orozco-Zitli. Hasta el momento sólo existen dos artículos publicados respecto al tema ([1] y [7]) y uno más que fue enviado recientemente para su publicación [5], en el cual aparecen nuevos resultados, productos de la investigación que desarrolle en mis estudios de maestría.

En particular, en [5], se introduce por primera vez en la literatura el concepto de Triodo Fuerte, el cual generaliza el concepto de triodo. En este proyecto pretendemos hacer un estudio más detallado respecto a este concepto ya que, como se menciono, recientemente ha sido definido.

Para la segunda parte de este proyecto nos enfocamos al estudio de los subcontinuos T-cerrados y T_n-cerrados (donde T denota la función T de Jones, ver Capítulo 3 de [10]). De forma general, decimos que un subconjunto A de un continuo X es T-cerrado si $T(A) = A$; y si n es un número natural, decimos que A es T_n-cerrado, si $T_n(A) = T_{n+1}(A)$ y $T_k(A) \neq T_{k+1}(A)$ para toda $k < n$. La definición de ser T-cerrado aparece en [2], y el concepto de ser T_n-cerrado la introducimos en este protocolo. En el 49th Spring Topology and Dynamics Conference 2015, el Dr. David P. Bellamy, propone estudiar la estructura de un continuo que contenga sólo un continuo T-cerrado. De forma general en este trabajo proponemos estudiar esta propiedad, así como estudiar la estructura de un continuo que contenga una cantidad finita o numerable de subcontinuos T_n-cerrados.

OBJETIVOS

El objetivo de la primera parte de este proyecto es estudiar el concepto de triodo fuerte para poder caracterizar el interior de los triodos bajo funciones suprayectivas en el espacio de representación de continuos y en la segunda parte del proyecto consiste en un estudio minucioso de los continuos que tienen una cantidad finita o numerable de continuos T_n-cerrados. En particular vamos a trabajar los siguientes problemas.

Problema 1. Caracterizar los triodos fuertes en la familia de los continuos localmente conexos.

Problema 2. Si X es un triodo y no es triodo fuerte, entonces X tiene al menos dos subcontinuos indescomponibles.

Algunos avances que tenemos son los siguientes. Con respecto al Problema 1, podemos mostrar que si X es un continuo localmente conexo y no es un triodo, entonces la dimensión de X es 1. Con esto, pretendemos mostrar que X es una gráfica finita, lo cual reduce el análisis a un argumento combinatorio, para así tener la clasificación deseada. Para resolver el Problemas 2, conjeturamos que basta con mostrar que si X es un continuo hereditariamente descomponible y p es un punto de X, entonces existe un subcontinuo propio que lo tiene en su interior. La solución de los Problemas 1, y 2 nos daría un estudio detallado de los triodos fuertes y a su vez resolverían el problema de caracterizar en interior de los triodos bajo funciones suprayectivas (Problema 3.16 de [1]). Los objetivos que pretendemos con respecto a la segunda parte de este proyecto es hacer un estudio minucioso de los continuos que tienen una cantidad finita o numerable de continuos T_n-cerrados. En este sentido, tenemos las siguientes conjeturas.

Conjetura 1. Si X contiene un único subcontinuo T-cerrado, entonces X contiene al menos un subcontinuo indescomponible.

Conjetura 2. Si X contiene un subcontinuo que es T_n-cerrado, entonces X contiene al menos n subcontinuos indescomponibles.

También vamos a estudiar los siguientes hiperespacios, que se definen por primera vez en este proyecto de investigación.

$$HT(X) = \{A \in C(X) : A \ es \ T - cerrado\}$$

y

$$HT_n(X) = \{A \in C(X) : A \ es \ T_k - cerrado \ con \ k \leq n\}.$$

Algunos resultados que tenemos al momento son los siguientes.

Teorema 1. Si X es localmente conexo, entonces $HT(X) = HT_n(X) = C(X)$.

Teorema 2. Si X es indescomponible, entonces $HT(X) = HT_n(X) = X$.

En particular conjeturamos que los regresos de los Teoremas 1 y 2 son verdad. También consideramos que el estudio de estos nuevos hiperespacios nos va a proporcionar herramienta nueva para identificar subcontinuos indescomponibles en un continuo; así como atacar propiedades de idempotencia y continuidad de la función T de Jones, y problemas de interés general.

METODOLOGÍA

Buscaremos bibliografía adecuada referente continuos indescomponibles y respecto a la función T de Jones, que nos permita conocer los avances recientes sobre los temas de interés. Posteriormente, leeremos y expondremos la bibliografía necesaria para el trabajo en un seminario con los directores del proyecto. Al mismo tiempo abordaremos lo problemas descritos en este proyecto y plantearemos nuevos problemas y líneas de investigación y escribiremos los resultados que se obtengan. Finalmente se revisarán los resultados obtenidos y se enviaran para su publicación en una revista indizada.

CRONOGRAMA DE ACTIVIDADES.

-Primer semestre.

Búsqueda de la Bibliografía necesaria para el desarrollo de la investigación. Lectura y exposición de los artículos en un seminario con los directores del proyecto. Se escribirá el trabajo realizado y las nuevas líneas de investigación en el procesador de textos científicos LaTex. Se tomaran los cursos Temas Selectos de Teoría de Hiperespacios y Temas Selectos de Teoría de Continuos.

- Segundo semestre.

Búsqueda de la Bibliografía necesaria para el desarrollo de la investigación. Se comenzará a resolver los problemas y conjeturas planteados en la primera parte de este proyecto. Se participará en un seminario con los directores del proyecto. Se escribirán los resultados obtenidos en el procesador de textos científicos LaTex.

- Tercer semestre.

Búsqueda de la Bibliografía necesaria para el desarrollo de la investigación. Se seguirá intentando resolver los problemas y conjeturas planteados en la primera parte de este proyecto. Se participará en un seminario con los directores del proyecto. Se escribirán los resultados obtenidos en el procesador de textos científicos LaTex y se enviará para su publicación a una revista indizada.

- Cuarto semestre.

Se seguirá trabajando en los problemas y conjeturas planteados en la primera y segunda parte de este proyecto. Se participará en un seminario con los directores del proyecto. Se escribirán los resultados obtenidos en el procesador de textos científicos LaTex. Se atenderán las sugerencias al artículo enviado.

- Quinto semestre.

Se seguirá trabajando en los problemas y conjeturas planteados en la primera parte y segunda de este proyecto. Se participará en un seminario con los directores del proyecto. Se escribirán los resultados obtenidos en el procesador de textos científicos LaTex y se revisarán los resultados obtenidos.

- Sexto semestre.

Se seguirá intentando resolver los problemas y conjeturas plateados en la primera parte y segunda de este proyecto. Se participará en un seminario con los directores del proyecto. Se escribirán los resultados obtenidos en el procesador de textos científicos LaTex y se publicarán en una revista indizada.

PRODUCTOS

Se publicarán dos artículos en revistas indizadas.

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Artículos

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Some aspects related to the Jones' set function $\mathcal{T}$

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A R T I C L E I N F O

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Unicoherent continuum

A B S T R A C T

We study three different topics related to the Jones' set function $\mathcal{T}$. The first topic is idempotency; we study differences between idempotency on continua and idempotency on closed sets. The second aspect that we present are some properties about the set $\mathcal{T}(2^X)$. Particularly, we show that it is not possible to find a continuum X such that $\mathcal{T}(2^X)$ is compact and countable; and we give a continuum X such that $\mathcal{T}(2^X)$ is countable. Finally, the third topic is the behavior of $\mathcal{T}$ on products. One of our main results is that the compactness of $\mathcal{T}(2^{X\times Y})$ implies the local connectedness of $X \times Y$.

1. Introduction

F. Burton Jones [4] defined the set function $\mathcal{T}$ in 1948. Later, in 1955, he [5] used the set function $\mathcal{T}$ to prove that homogeneous decomposable continua admit continuous decompositions into homogeneous continua whose quotient space is an aposyndetic homogeneous continuum. Since then many papers have been written with results using the set function $\mathcal{T}$, see [8, Chapter 3].

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The paper is divided in five sections. After the preliminaries section, in Section 3, we study the idempotency of $\mathcal{T}$ on continua. Example 3.1 gives an example of a continuum X such that $\mathcal{T}$ is idempotent on continua but it is not idempotent on closed sets. We present some classes of continua such that the idempotency of $\mathcal{T}$ on continua implies the idempotency of $\mathcal{T}$ on closed sets (Theorem 3.3 and Corollary 3.4). We also prove that if $\mathcal{T}$ is idempotent on continua, then $\mathcal{T}^{2n}(A) = \mathcal{T}^{2n-1}(A)$ for each element A of the n-fold hyperspace of the continuum (Theorem 3.6). In Section 4, we continue the study of images of $\mathcal{T}$ [3]. Inspired by [1, Theorem 3.1], we prove that $\mathcal{T}(2^X)$ is an analytic set for each continuum X (Theorem 4.1). We also show that for a continuum X, if $\mathcal{T}(2^X)$ is compact, then $\mathcal{T}(2^X)$ is either finite or uncountable (Theorem 4.11). We present an example of a continuum X such that $\mathcal{T}(2^X)$ is countable (Example 4.14). In Section 5, we consider the set function $\mathcal{T}$ defined on the product of two continua. First, we consider the family of $\mathcal{T}$-closed sets and prove that if the family of $\mathcal{T}$-closed sets of the product of two continua X and Y is compact, then $X \times Y$ is locally connected (Lemma 5.2). Then we give a partial positive answer to [8, Question 9.2.10] by showing that if $\mathcal{T}$ is defined on a product of two continua, then the continuity of $\mathcal{T}$ restricted to the hyperspace of subcontinua of such product implies the continuity of $\mathcal{T}$ (Theorem 5.3). Also, we demonstrate that the compactness of $\mathcal{T}(2^{X\times Y})$, where X and Y are continua, implies the local connectedness of $X \times Y$ (Theorem 5.7).

2. Preliminaries

If (Z, d) is a metric space, then given a point z in Z and a number $\varepsilon > 0$, we let $\mathcal{V}_\varepsilon(z) = \{z' \in Z \mid d(z, z') < \varepsilon\}$. If A is a subset of Z, then the *interior*, *closure* and *boundary* of A are denoted by $\mathrm{Int}_Z(A)$, $\mathrm{Cl}_Z(A)$ and $\mathrm{Bd}_Z(A)$, respectively. We omit the subscripts if there is no confusion. The *power set of* Z is denoted by $\mathcal{P}(Z)$. The *cardinality of a set* A is denoted by $|A|$.

A *map* is a continuous function. A topological space X is called *completely metrizable* if it admits a compatible metric d such that (X, d) is complete. A separable completely metrizable space is called a *Polish space*. A subset A of X is called *analytic* if there exist a Polish space Y and a map $f : Y \to X$ with $f(Y) = A$.

A *compactum* is a nonempty compact metric space. A *continuum* is a connected compactum. A *subcontinuum* is a continuum contained in some metric space. A continuum X is said to be *unicoherent* provided that whenever A and B are subcontinua of X such that $X = A \cup B$, then $A \cap B$ is connected. A continuum is *hereditarily unicoherent* if each of its subcontinua is unicoherent. A continuum X is called *decomposable* provided that there exist two proper subcontinua A and B of X such that $X = A \cup B$. If a continuum is not decomposable, we say that it is *indecomposable*.

Let X be a continuum. We say that X is *irreducible* provided that there exist two points a and b in X, for which there is not a proper subcontinuum Y of X such that $\{a, b\} \subseteq Y$.

Given a continuum X, we consider some of its hyperspaces:

- $2^X = \{A \subseteq X : A \text{ is closed and nonempty}\}$;
- $\mathcal{C}_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}, n \in \mathbb{N}$;
- $\mathcal{F}_n(X) = \{A \subseteq X : 1 \le |A| \le n\}, n \in \mathbb{N}$;
- $\mathcal{F}_\infty(X) = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n(X)$, $\mathcal{C}(X) = \mathcal{C}_1(X)$; and
- $\mathcal{C}_\infty(X) = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n(X)$.

For each $A, B \in 2^X$, let

$$\mathcal{H}(A, B) = \inf\{r > 0 : A \subseteq \mathcal{V}_r(B) \text{ and } B \subseteq \mathcal{V}_r(A)\},$$

where $\mathcal{V}_s(D) = \{x \in X : d(x, z) < s, \text{ for some } z \in D\}$, $D \in 2^X$ and $s > 0$. $\mathcal{H}$ is the Hausdorff metric on 2^X. Note that $\mathcal{C}(X) \subseteq \mathcal{C}_n(X) \subseteq \mathcal{C}_\infty(X) \subseteq 2^X$ and $\mathcal{F}_n(X) \subseteq \mathcal{F}_\infty(X) \subseteq 2^X$. It is known that 2^X is a continuum

[8, 1.8.9]. Also, it is well known that $\mathcal{C}(X)$, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are continua, for every positive number n. It is easy to see that the hyperspace $\mathcal{F}_\infty(X)$ is dense in 2^X. Since $\mathcal{F}_\infty(X) \subseteq \mathcal{C}_\infty(X)$, $\mathcal{C}_\infty(X)$ is also a dense subset of 2^X.

Given a finite family, $U_1, \ldots, U_m$, of nonempty subsets of X, we define $\langle U_1, \ldots, U_m \rangle$ as the set:

$$\left\{ A \in 2^X : A \subseteq \bigcup_{k=1}^{m} U_k \text{ and } A \cap U_k \neq \emptyset \text{ for each } k \in \{1, \ldots, m\} \right\}.$$

It is known that the family of all subsets of 2^X of the form $\langle U_1, \ldots, U_m \rangle$, where $U_1, \ldots, U_m$ are nonempty open subsets of X, form a basis for a topology for 2^X called the *Vietoris Topology* [9, (0.11)]. Note that the Vietoris Topology and the Topology induced by the Hausdorff metric coincide [9, (0.13)].

Given a compactum X, we define $\mathcal{T}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\mathcal{T}(A) = X \setminus \{x \in X : \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in \operatorname{Int}(W) \subseteq W \subseteq X \setminus A\},$$

for each $A \in \mathcal{P}(X)$. The function $\mathcal{T}$ is called *Jones' set function* $\mathcal{T}$.

Note that $\mathcal{T}(A)$ is closed, for each $A \in \mathcal{P}(X)$. Thus, the restriction $\mathcal{T}|_{2^X} : 2^X \to 2^X$ is well defined. When we say that $\mathcal{T}$ is continuous, we refer to $\mathcal{T}|_{2^X}$. A continuum X is $\mathcal{T}$*-additive* provided that for each $A, B \in 2^X$, $\mathcal{T}(A \cup B) = \mathcal{T}(A) \cup \mathcal{T}(B)$. Moreover, X is called $\mathcal{T}$*-symmetric* if for each $A, B \in 2^X$, we have that

$$A \cap \mathcal{T}(B) \neq \emptyset \text{ if and only if } \mathcal{T}(A) \cap B \neq \emptyset.$$

The function $\mathcal{T}$ is said to be:

- *idempotent* provided that $\mathcal{T}^2(A) = \mathcal{T}(A)$, for each $A \in \mathcal{P}(X)$;
- *idempotent on closed sets* provided that $\mathcal{T}^2(A) = \mathcal{T}(A)$, for each $A \in 2^X$; and
- *idempotent on continua* provided that $\mathcal{T}^2(A) = \mathcal{T}(A)$, for each $A \in \mathcal{C}(X)$.

Remark 2.1. Note that if X is a compactum, then:

1. if $\mathcal{T}$ is idempotent, then $\mathcal{T}$ is idempotent on closed sets;
2. if $\mathcal{T}$ is idempotent on closed sets, then $\mathcal{T}$ is idempotent on continua.

Observe that for the suspension over the harmonic sequence $\{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$, $\mathcal{T}$ is idempotent on closed sets, but it is not idempotent. In Section 3, we study relationships between idempotency on continua and idempotency on closed sets.

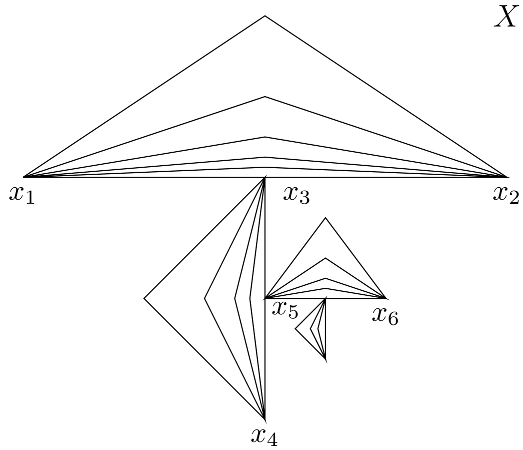
3. Idempotency of $\mathcal{T}$ on continua

We study the idempotency of the set function $\mathcal{T}$ on continua. We give some classes of continua for which the idempotency of $\mathcal{T}$ on continua implies the idempotency of $\mathcal{T}$ on closed sets.

The following example shows that the converse of Remark 2.1 (2) is not true in general.

Example 3.1. There exists a continuum X such that $\mathcal{T}$ is idempotent on continua, and $\mathcal{T}$ is not idempotent on closed sets.

Let X be the continuum defined by the closure of the union of a sequence $\{X_n\}_{n=1}^{\infty}$ of harmonic suspensions such that the size of X_{n+1} is $\frac{1}{2}$ the size of X_n. Also, one of the vertices of X_2 is the mid point of the limit segment of X_1, one of the vertices of X_3 is the mid point of the limit segment of X_2, etc. as shown in the following picture:



Note that $\mathcal{T}$ is idempotent on continua. The limit segment in X_n is denoted by $\overline{x_{2n-1}x_{2n}}$, for each $n \in \mathbb{N}$. Also, observe that given $n \in \mathbb{N}$:

- $\mathcal{T}(\{x_1, x_2, x_4, x_6, \ldots, x_{2n}\}) = \overline{x_1x_2} \cup \{x_4, \ldots, x_{2n}\}$;
- $\mathcal{T}^2(\{x_1, x_2, x_4, x_6, \ldots, x_{2n}\}) = \overline{x_1x_2} \cup \overline{x_3x_4} \cup \{x_6, \ldots, x_{2n}\}$;

  $\vdots$
- $\mathcal{T}^n(\{x_1, x_2, x_4, x_6, \ldots, x_{2n}\}) = \overline{x_1x_2} \cup \overline{x_3x_4} \cup \ldots \cup \overline{x_{2n-1}x_{2n}}$; and
- $\mathcal{T}^{n+1}(\{x_1, x_2, x_4, x_6, \ldots, x_{2n}\}) = \mathcal{T}^n(\{x_1, x_2, x_4, x_6, \ldots, x_{2n}\})$.

Therefore, $\mathcal{T}$ is not idempotent on closed sets.

A natural question is the following.

Question 3.2. What kind of continua have the property that the idempotency on continua implies the idempotency on closed sets?

We give a partial answer to Question 3.2 in the following theorem.

Theorem 3.3. *Let X be a $\mathcal{T}$-additive continuum. If $\mathcal{T}$ is idempotent on continua, then $\mathcal{T}$ is idempotent on closed sets.*

Proof. Let A be a nonempty closed subset of X. Since X is $\mathcal{T}$-additive, by [8, Corollary 3.1.51], $\mathcal{T}(A) = \bigcup\{\mathcal{T}(\{a\}) \mid a \in A\}$. Note that $\mathcal{T}(\{x\})$ is a continuum [8, Theorem 3.1.21] for all $x \in X$. Hence, $\mathcal{T}^2(A) = \mathcal{T}\left(\bigcup\{\mathcal{T}(\{a\}) \mid a \in A\}\right) = \bigcup\{\mathcal{T}^2(\{a\}) \mid a \in A\} = \bigcup\{\mathcal{T}(\{a\}) \mid a \in A\} = \mathcal{T}(A)$. The second equality is by $\mathcal{T}$-additivity [8, Theorem 3.1.48] and the third equality is by the idempotency of $\mathcal{T}$ on continua. □

Corollary 3.4. *If X is either an irreducible or a hereditarily unicoherent continuum, then the idempotency on continua implies the idempotency on closed sets.*

Proof. Suppose X is irreducible. Then X is $\mathcal{T}$-symmetric [8, Corollary 3.1.42]. Hence, X is $\mathcal{T}$-additive [8, Theorem 3.1.49]. If X is hereditarily unicoherent, then X is $\mathcal{T}$-additive [8, Theorem 3.1.50]. In both cases, the corollary now follows from Theorem 3.3. □

The following proposition generalizes [8, Corollary 3.1.78].

Proposition 3.5. *Let X be a continuum and let $n \in \mathbb{N}$. If $A = A_1 \cup \cdots \cup A_n \in \mathcal{C}_n(X)$, where $A_1, \dots, A_n$ are the components of A, is such that $\mathcal{T}(A) \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$, then $\mathcal{T}(A) = \mathcal{T}(A_1) \cup \cdots \cup \mathcal{T}(A_n)$.*

Proof. Suppose that $\mathcal{T}(A) = D_1 \cup \cdots \cup D_n$, where $D_1, \dots, D_n$ are the components of $\mathcal{T}(A)$. Since $A \subseteq D_1 \cup \cdots \cup D_n$ and $D_i \cap A \neq \emptyset$ for each $i \in \{1, \dots, n\}$ [8, Corollary 3.7.4], we may assume that $A_i \subseteq D_i$ for each $i \in \{1, \dots, n\}$. Observe that $\mathcal{T}(A_i) = \mathcal{T}(A \cap D_i) = D_i$ [8, Corollary 3.1.75] for each $i \in \{1, \dots, n\}$. Therefore, $\mathcal{T}(A) = \mathcal{T}(A_1) \cup \cdots \cup \mathcal{T}(A_n)$. □

Theorem 3.6. *Let X be a continuum and let $n \in \mathbb{N}$. If $\mathcal{T}$ is idempotent on continua, then $\mathcal{T}^{2n}(A) = \mathcal{T}^{2n-1}(A)$ for each $A \in \mathcal{C}_n(X)$.*

Proof. We do the proof by induction over n. Since $\mathcal{T}$ is idempotent on continua, $\mathcal{T}^2(A) = \mathcal{T}(A)$ for each $A \in \mathcal{C}(X)$. Suppose that $\mathcal{T}^{2k-2}(A) = \mathcal{T}^{2k-3}(A)$ for each $A \in \mathcal{C}_{k-1}(A)$. Let $B \in \mathcal{C}_k(X)$. We show that $\mathcal{T}^{2k}(B) = \mathcal{T}^{2k-1}(B)$.

By [8, Corollary 3.1.76], $\mathcal{T}(B) \in \mathcal{C}_k(X)$. Note that if $\mathcal{T}(B) \in \mathcal{C}_{k-1}(X)$, then $\mathcal{T}^{2k-2}(\mathcal{T}(B)) = \mathcal{T}^{2k-3}(\mathcal{T}(B))$, and $\mathcal{T}^{2k-1}(B) = \mathcal{T}^{2k-2}(B)$. Hence, suppose that $\mathcal{T}(B) \in \mathcal{C}_k(X) \setminus \mathcal{C}_{k-1}(X)$. By Proposition 3.5, $\mathcal{T}(B) = \mathcal{T}(B_1) \cup \cdots \cup \mathcal{T}(B_k)$, where $B_1, \dots, B_k$ are the components of B. By [8, Corollary 3.1.76], $\mathcal{T}^2(B) \in \mathcal{C}_k(X)$. We consider two cases:

Case 1. $\mathcal{T}^2(B) \in \mathcal{C}_{k-1}(X)$.

Then $\mathcal{T}^{2k-2}(\mathcal{T}^2(B)) = \mathcal{T}^{2k-3}(\mathcal{T}^2(B))$ and hence, $\mathcal{T}^{2k}(B) = \mathcal{T}^{2k-1}(B)$.

Case 2. $\mathcal{T}^2(B) \in \mathcal{C}_k(X) \setminus \mathcal{C}_{k-1}(X)$.

By Proposition 3.5, $\mathcal{T}^2(B) = \mathcal{T}(\mathcal{T}(B_1) \cup \cdots \cup \mathcal{T}(B_k)) = \mathcal{T}^2(B_1) \cup \cdots \cup \mathcal{T}^2(B_k)$. Since $\mathcal{T}$ is idempotent on continua, $\mathcal{T}^2(B) = \mathcal{T}(B)$. Therefore, $\mathcal{T}^{2k}(B) = \mathcal{T}^{2k-1}(B)$ for every $B \in \mathcal{C}_k(X)$. □

Remark 3.7. Note that Example 3.1 shows that Theorem 3.6 cannot be improved to obtain a global bound for all the elements of $\mathcal{C}_\infty(X)$. Also in the same example, observe that if $A = \{x_1, p\} \cup \{x_{2n} : n \in \mathbb{N}\}$, where $\lim_{n\to\infty} x_{2n} = p$, then $\mathcal{T}^k(A) \neq \mathcal{T}^{k+1}(A)$ for any $k \in \mathbb{N}$.

Question 3.8. Let X be a continuum for which there exists $m \in \mathbb{N}$ such that $\mathcal{T}^m(A) = \mathcal{T}^{m-1}(A)$ for each $A \in \mathcal{C}_\infty(X)$. Then does it follow that $\mathcal{T}^m(B) = \mathcal{T}^{m-1}(B)$ for every $B \in 2^X$?

Compare the following result with [1, Theorem 3.6].

Theorem 3.9. *Let X be a continuum and let $A \in 2^X$ be such that $\mathcal{T}(A) = A$. If B is a closed subset of X that is the union of some components of A, then $\mathcal{T}(B) = B$.*

Proof. Suppose that $B = \bigcup_{\lambda\in\Lambda} K_\lambda$, where K_λ is a component of A, for each $\lambda \in \Lambda$. We know that $B \subseteq \mathcal{T}(B)$. We shall prove that $\mathcal{T}(B) \subseteq B$. Let L be a component of $\mathcal{T}(B)$. By [8, Corollary 3.7.4], there exists $\lambda_0 \in \Lambda$ such that $L \cap K_{\lambda_0} \neq \emptyset$. Since $\mathcal{T}(B) \subseteq \mathcal{T}(A)$ and $\mathcal{T}(A) = A$, we have that $L \cup K_{\lambda_0}$ is a subcontinuum of A. Hence, $L \cup K_{\lambda_0} \subseteq K_{\lambda_0}$ because K_{λ_0} is a component. Since L was arbitrary, this show that $\mathcal{T}(B) \subseteq B$. Therefore, $\mathcal{T}(B) = B$. □

4. Images of $\mathcal{T}$

We continue the study of the properties of $\mathcal{T}(2^X)$ started in [3]. We begin by showing that $\mathcal{T}(2^X)$ is an analytic set for every continuum X. We show that if X is a continuum such that $\mathcal{T}(2^X)$ is compact, then $\mathcal{T}(2^X)$ is either finite or uncountable. We give an example of a continuum X such that $\mathcal{T}(2^X)$ is countable.

Theorem 4.1. *The set $\mathcal{T}(2^X)$ is an analytic set, for any continuum X.*

Proof. Let $\mathcal{F} = \{(A, B) \in 2^X \times 2^X \mid \mathcal{T}(A) = B\}$. We prove that $\mathcal{F}$ is a G_δ-set. Let $M = \{(A, B) \in 2^X \times 2^X \mid B \subseteq \mathcal{T}(A)\}$ and $N = \{(A, B) \in 2^X \times 2^X \mid \mathcal{T}(A) \subseteq B\}$.

Claim 4.2. *M is a closed subset of $2^X \times 2^X$.*

Let $(C, D) \notin M$. Hence, there exists $x \in D \setminus \mathcal{T}(C)$. Since X is regular, there exist open sets U and V such that $x \in U$, $\mathcal{T}(C) \subseteq V$ and $U \cap V = \emptyset$. Since $\mathcal{T}$ is upper semicontinuous [8, Theorem 3.3.1], there exists an open set $\mathcal{U} \subseteq 2^X$ such that $C \in \mathcal{U}$ and $\mathcal{T}(E) \subseteq V$ for each $E \in \mathcal{U}$. Let $\mathcal{W} = \mathcal{U} \times \langle X, U\rangle$. Observe that $(C, D) \in \mathcal{W}$ and $\mathcal{W}$ is an open subset of $2^X \times 2^X$. Furthermore, if $(E, F) \in \mathcal{W}$, then $\mathcal{T}(E) \subseteq V$ and $F \cap U \neq \emptyset$. Thus, $F \setminus \mathcal{T}(E) \neq \emptyset$ and $(E, F) \notin M$. Therefore, $\mathcal{W} \subseteq (2^X \times 2^X) \setminus M$ and M is closed.

Claim 4.3. *N is a G_δ-set.*

For each $n \in \mathbb{N}$, let

$$\mathcal{L}_n = \mathrm{Cl}_{2^X \times 2^X}(\{(A, B) \in 2^X \times 2^X \mid \mathcal{V}_{\frac{1}{n}}(a) \cap B = \emptyset, \text{ for some } a \in \mathcal{T}(A)\}).$$

We see that $N = (2^X \times 2^X) \setminus \bigcup_{n\in\mathbb{N}} \mathcal{L}_n$. Let $(A, B) \notin N$. Then there exists $x \in \mathcal{T}(A) \setminus B$. Let $n_0 \in \mathbb{N}$ be such that $\mathcal{V}_{\frac{1}{n_0}}(x) \cap B = \emptyset$. Thus, $(A, B) \in \mathcal{L}_{n_0} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{L}_n$. Therefore, $(2^X \times 2^X) \setminus \bigcup_{n\in\mathbb{N}} \mathcal{L}_n \subseteq N$. Conversely, suppose that $(A, B) \in \mathcal{L}_n$, for some $n \in \mathbb{N}$. Let $\{(A_m, B_m)\}_{m\in\mathbb{N}}$ be a sequence in $\{(A, B) \in 2^X \times 2^X \mid \mathcal{V}_{\frac{1}{n}}(a) \cap B = \emptyset$, for some $a \in \mathcal{T}(A)\}$, such that $\lim_{m\to\infty}(A_m, B_m) = (A, B)$.

For each $m \in \mathbb{N}$, let $a_m \in \mathcal{T}(A_m)$ be such that $\mathcal{V}_{\frac{1}{n}}(a_m) \cap B_m = \emptyset$. Since X is compact, there exists a subsequence $\{a_{m_k}\}_{k\in\mathbb{N}}$ of $\{a_m\}_{m\in\mathbb{N}}$ such that $\lim_{k\to\infty} a_{m_k} = a_0$, for some $a_0 \in X$. Since $\mathcal{T}$ is upper semicontinuous [8, Theorem 3.3.1], $\limsup \mathcal{T}(A_{m_k}) \subseteq \mathcal{T}(A)$. Thus, $a_0 \in \mathcal{T}(A)$. Suppose that $\mathcal{V}_{\frac{1}{2n}}(a_0) \cap B \neq \emptyset$. Since $\lim_{k\to\infty} a_{m_k} = a_0$ and $\lim_{k\to\infty} B_{m_k} = B$, we have that there exists $l \in \mathbb{N}$ such that $a_{m_l} \in \mathcal{V}_{\frac{1}{2n}}(a_0)$ and $B_{m_l} \cap \mathcal{V}_{\frac{1}{2n}}(a_0) \neq \emptyset$. Let $z \in B_{m_l} \cap \mathcal{V}_{\frac{1}{2n}}(a_0)$. Note that $d(a_{m_l}, z) \leq d(a_{m_l}, a_0) + d(a_0, z) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$. Hence, $\mathcal{V}_{\frac{1}{n}}(a_{m_l}) \cap B_{m_l} \neq \emptyset$. A contradiction. Thus, $\mathcal{V}_{\frac{1}{2n}}(a_0) \cap B = \emptyset$ and $a_0 \in \mathcal{T}(A) \setminus B$. Therefore, $(A, B) \notin N$, and hence, N is a G_δ-set.

Note that $\mathcal{F} = M \cap N$. Thus, $\mathcal{F}$ is a G_δ-set. By [6, Theorem 3.11], $\mathcal{F}$ is a Polish space. Furthermore, $\mathcal{T}(2^X) = \pi_2(\mathcal{F})$, where $\pi_2 \colon 2^X \times 2^X \to 2^X$ is defined by $\pi_2(A, B) = B$, for each $(A, B) \in 2^X \times 2^X$. Therefore, $\mathcal{T}(2^X)$ is an analytic set. □

Note that if $\mathcal{T}$ is idempotent on closed sets, then $\mathcal{T}(2^X) = \{A \in 2^X : \mathcal{T}(A) = A\}$. Thus, the next corollary follows from [1, Theorem 3.1].

Corollary 4.4. *Let X be a continuum. If $\mathcal{T}$ is idempotent on closed sets, then $\mathcal{T}(2^X)$ is a G_δ subset of 2^X.*

The class of Borel sets of X is the σ-algebra generated by the open sets of X. Hence, if $\mathcal{T}$ is idempotent on closed sets, then $\mathcal{T}(2^X)$ is a Borel set, Corollary 4.4. Naturally, we have the following question.

Question 4.5. Given a continuum X, is $\mathcal{T}(2^X)$ a Borel set of 2^X?

Notation 4.6. Given X a continuum and $x \in X$, we define $L_x = \{z \in X \mid \mathcal{T}(\{z\}) = \mathcal{T}(\{x\})\}$.

The proof of the following proposition can be found in [3, Proposition 3.10].

Proposition 4.7. *Let X be a continuum. If $\mathcal{T}(2^X)$ is a countably infinite set, then there exists a family of indecomposable continua $\{M_n \mid n \in \mathbb{N}\}$, such that:*

1. $X = \mathrm{Cl}_X(\bigcup_{n\in\mathbb{N}} M_n)$,
2. *for each $n \in \mathbb{N}$, there exists $x_n \in M_n$ satisfying:*
   (a) $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_X(L_{x_n})$ *is dense in X,*
   (b) $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) \neq \emptyset$,
   (c) *M_n is irreducible about L_{x_n},*
   (d) $L_{x_n} \subseteq \mathrm{Cl}_X(L_{x_n}) \subseteq M_n \subseteq \mathcal{T}(\{x_n\})$, *and*
   (e) $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) \subseteq X \setminus \mathrm{Cl}_X(\bigcup_{k\in\mathbb{N}\setminus\{n\}} M_k)$.

Compare the following result with *2. (a)* in Proposition 4.7.

Lemma 4.8. *Let X be a continuum. If $\mathcal{T}(2^X)$ is a countably infinite set, then $\bigcup_{n\in\mathbb{N}} \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n}))$ is dense in X.*

Proof. Let U be a nonempty open subset of X. Since $\mathcal{T}(\mathcal{F}_1(X))$ is countably infinite, there exists a countable set S, such that $\mathcal{T}(\mathcal{F}_1(X)) = \{\mathcal{T}(\{z_j\}) \mid z_j \in X, j \in S\}$, where $\{x_n \mid n \in \mathbb{N}\} \subseteq \{z_j \mid j \in S\}$. Observe that by the proof of [3, Theorem 3.2],

$$\mathrm{Int}_X(\mathrm{Cl}_X(L_{z_j})) \neq \emptyset \text{ if and only if } z_j = x_k, \text{ for some } k \in \mathbb{N}. \tag{4.1}$$

It is clear that $X = \bigcup_{j\in S} \mathrm{Cl}_X(L_{z_j})$. Thus, $U = \bigcup_{j\in S}(U \cap \mathrm{Cl}_X(L_{z_j}))$. Since U is a Baire space [10, 25.3], there exists $j_0 \in S$ such that $U \cap \mathrm{Cl}_X(L_{z_{j_0}})$ has nonempty interior. Hence, $U \cap \mathrm{Int}_X(\mathrm{Cl}_X(L_{z_{j_0}})) \neq \emptyset$. Therefore, there exists $k \in \mathbb{N}$ such that $U \cap \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_k})) \neq \emptyset$, by (4.1), and $\bigcup_{n\in\mathbb{N}} \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n}))$ is dense in X. □

Lemma 4.9. *Let X be a continuum such that $\mathcal{T}(2^X)$ is countably infinite. Let $\{M_n \mid n \in \mathbb{N}\}$ be a family of indecomposable continua satisfying Proposition 4.7. If K is an indecomposable subcontinuum of X such that $L_{x_i} \subseteq K$, for some $i \in \mathbb{N}$, then $K \subseteq \mathcal{T}(\{x_i\})$.*

Proof. Let $i \in \mathbb{N}$ be such that $L_{x_i} \subseteq K$. Suppose that $K \setminus \mathcal{T}(\{x_i\}) \neq \emptyset$. Let $y \in K \setminus \mathcal{T}(\{x_i\})$. Then there exists a continuum H such that $y \in \mathrm{Int}_X(H)$ and $x_i \notin H$. Note that H intersects all the composants of K.

We show that $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_i})) \setminus H \neq \emptyset$. To this end, suppose that $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_i})) \subseteq H$. Hence, $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_i})) \subseteq \mathrm{Int}_X(H)$. Thus, $\mathrm{Cl}_X(L_{x_i}) \cap \mathrm{Int}_X(H) \neq \emptyset$ and $L_{x_i} \cap \mathrm{Int}_X(H) \neq \emptyset$. This contradicts the fact that $x_i \notin H$. Therefore, $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_i})) \setminus H \neq \emptyset$.

Let $U = \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_i})) \setminus H$. Since $L_{x_i} \subseteq K$, $U \subseteq K$. For each $x \in U$, let Q_x be the component of $K \setminus \mathrm{Int}_X(H)$ such that $x \in Q_x$. Observe that $Q_x \cap H \neq \emptyset$, by [9, Theorem 20.2].

Claim 4.10. *Let z and w be points of U such that z and w belong to different composants of K. Then there exists a continuum W such that $w \in \mathrm{Int}_X(W)$ and $z \notin W$.*

Let $\varepsilon > 0$ be such that $z \notin \mathcal{V}_\varepsilon(w)$ and $\mathcal{V}_\varepsilon(w) \subseteq U$. Let $W_\varepsilon = \mathrm{Cl}_X(\bigcup\{Q_x \mid x \in \mathcal{V}_\varepsilon(w)\}) \cup H$. Then W_ε is a continuum and $w \in \mathrm{Int}_X(W_\varepsilon)$. We show that there exists $\delta > 0$, such that $z \notin W_\delta$. Suppose that for each $n \in \mathbb{N}$, $z \in W_{\frac{1}{n}}$. Since $z \notin H$, there exists $r_n \in \mathcal{V}_{\frac{1}{n}}(z)$ such that $\min\{d(w,r) \mid r \in Q_{r_n}\} < \frac{1}{n}$. Note that

$\{Q_{r_n}\}_{n\in\mathbb{N}}$ is a sequence of $\mathcal{C}(K)$ such that $Q_{r_n} \cap \mathrm{Int}_X(H) = \emptyset$, for each $n \in \mathbb{N}$. Since $\mathcal{C}(K)$ is compact, $\{Q_{r_n}\}_{n\in\mathbb{N}}$ has a limit point Q in $\mathcal{C}(K)$, where $Q \cap \mathrm{Int}_X(H) = \emptyset$. Observe that $\{z, w\} \subseteq \liminf(Q_{r_n})$. Thus, $\{z, w\} \subseteq Q$ and Q is a proper subcontinuum of K. This contradicts the fact that z and w belong to different composants of K. Therefore, there exists $\delta > 0$, such that $z \notin W_\delta$. It is clear that $W = W_\delta$ satisfies conditions of Claim 4.10.

Now, observe that W is a continuum such that $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_i})) \cap \mathrm{Int}_X(W) \neq \emptyset$ and $(X \setminus W) \cap \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_i})) \neq \emptyset$. Thus, there exist p and q in L_{x_i} such that $p \in \mathrm{Int}_X(W)$ and $q \notin W$. This contradicts the fact that $\mathcal{T}(\{p\}) = \mathcal{T}(\{q\}) = \mathcal{T}(\{x_i\})$. Therefore, $K \setminus \mathcal{T}(\{x_i\}) = \emptyset$ and $K \subseteq \mathcal{T}(\{x_i\})$. □

Examples 3.19 and 4.7 of [3], show continua X such that $\mathcal{T}(2^X)$ is compact and not finite, respectively.

Theorem 4.11. *Let X be a continuum such that $\mathcal{T}(2^X)$ is compact. Then $\mathcal{T}(2^X)$ is a finite or an uncountable set.*

Proof. Suppose that $\mathcal{T}(2^X)$ is countably infinite. By [3, Theorem 3.5], $\mathcal{T}(\mathcal{F}_1(X))$ is also countably infinite. Thus, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ as in Proposition 4.7.

Claim 4.12. *There exist an infinite subset $\mathcal{N}$ of $\mathbb{N}$ and a sequence $\{z_i\}_{i\in\mathbb{N}} \subseteq X$ such that $z_i \in \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_{n_i}}))\cap L_{x_{n_i}}$ (Notation 4.6), for each $n_i \in \mathcal{N}$, $\lim_{i\to\infty} z_i = z_0$ and $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_{n_i}}))\cap\mathcal{T}(\{z_0\}) = \emptyset$ for every $n_i \in \mathcal{N}$.*

Since X is compact, without loss of generality, we may assume that there exists $y_n \in \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) \cap L_{x_n}$, for each $n \in \mathbb{N}$, such that $\lim_{n\to\infty} y_n = y_0$ and $\lim_{n\to\infty} \mathcal{T}(\{y_n\}) = L$, for some $L \in 2^X$. Since $\mathcal{T}$ is upper semicontinuous [8, Theorem 3.3.1], $L \subseteq \mathcal{T}(\{y_0\})$. If $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) \cap \mathcal{T}(\{y_0\}) = \emptyset$ for infinitely many n's, then we complete the proof of Claim 4.12. Hence, suppose that $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) \cap \mathcal{T}(\{y_0\}) \neq \emptyset$, for any $n \in \mathbb{N}$. Note that if $w \in \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n}))$ and K is a continuum such that $w \in \mathrm{Int}_X(K)$, then $L_{x_n} \subseteq K$. Also, $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) \subseteq K$ and $y_0 \in K$. Thus, $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) \subseteq \mathcal{T}(\{y_0\})$, for each $n \in \mathbb{N}$. Now, we see that $L \cap \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) = \emptyset$, for each $n \in \mathbb{N}$. Suppose that there exists $k \in \mathbb{N}$ such that $L \cap \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_k})) \neq \emptyset$. Since $\lim_{n\to\infty} \mathcal{T}(\{y_n\}) = L$, there exists $l \neq k$ such that $\mathcal{T}(\{y_l\})\cap\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_k})) \neq \emptyset$. Since $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_k})) \subseteq M_k$, $y_l \in M_k$. This contradicts the fact that $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_l})) \cap M_k = \emptyset$ (see *(e)* of Proposition 4.7). Therefore, $L \cap \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) = \emptyset$, for each $n \in \mathbb{N}$, and $L \subsetneq \mathcal{T}(\{y_0\})$.

Since $\mathcal{T}(2^X)$ is compact, $L \in \mathcal{T}(2^X)$. Hence, there exists $A \in 2^X, A \subseteq L$, such that $\mathcal{T}(A) = L$.

Since $\bigcup_{n\in\mathbb{N}} \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n}))$ is dense in X, Lemma 4.8, there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that, for each $n_i \in \mathcal{N}$, there exists $z_i \in \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_{n_i}})) \cap L_{x_{n_i}}$, where $\lim_{i\to\infty} z_i = z_0$ and $z_0 \in A$. Since $L \cap \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_n})) = \emptyset$, for each $n \in \mathbb{N}$, and $\mathcal{T}(\{z_0\}) \subseteq \mathcal{T}(A) = L$, we have that $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_{n_s}})) \cap \mathcal{T}(\{z_0\}) = \emptyset$, for every $n_s \in \mathcal{N}$.

Claim 4.13. *Let $P = \{z_i \mid i \in \mathbb{N}\} \cup \{z_0\} \in 2^X$. For each $i \in \mathbb{N}$, there exists a subcontinuum E_i of X such that $z_i \in \mathrm{Int}_X(E_i)$ and $E_i \cap P = \{z_i\}$.*

Let $i \in \mathbb{N}$. Since $z_i \in \mathrm{Int}_X(\mathrm{Cl}_X(L_{x_{n_i}}))$ and $\mathrm{Int}_X(\mathrm{Cl}_X(L_{x_{n_i}})) \cap \mathcal{T}(\{z_0\}) = \emptyset$, we have that $z_i \notin \mathcal{T}(\{z_0\})$. Thus, there exists a subcontinuum K of X such that $z_i \in \mathrm{Int}_X(K)$ and $z_0 \notin K$. Since $z_i \in \mathcal{T}(\{x_{n_i}\})$, $L_{x_{n_i}} \subseteq K$. By [10, Theorem 28.4], there exists a subcontinuum E_i of K such that E_i is irreducible about $L_{x_{n_i}}$. It is not difficult to see that E_i is indecomposable (see [3, Proposition 3.1]). Therefore, $E_i \subseteq \mathcal{T}(\{x_{n_i}\})$, Lemma 4.9, and $E_i \cap P = \{z_i\}$, for each $i \in \mathbb{N}$.

Let S and R be two different subsequences of P. If $z_j \in S\setminus R$, then $z_j \in \mathrm{Int}_X(E_j)$ and $E_j\cap(R\cup\{z_0\}) = \emptyset$. Thus, $z_j \in \mathcal{T}(S \cup \{z_0\}) \setminus \mathcal{T}(R \cup \{z_0\})$. Since P has uncountably many subsequences, $\mathcal{T}(2^X)$ is uncountable. A contradiction. Therefore, $\mathcal{T}(2^X)$ is either finite or uncountable. □

Example 4.14. There exists a continuum X such that $\mathcal{T}(2^X)$ is a countably infinite set.

Let $\{M_n \mid n \in \mathbb{N}\}$ be a sequence of indecomposable continua in $\mathbb{R}^2$ such that (i) $M_i \cap M_j = \emptyset$ for each $i \neq j$, and (ii) $\lim_{i\to\infty} M_i = \{p\}$, for some $p \in \mathbb{R}^2$. Let $Z = (\bigcup_{n\in\mathbb{N}} M_n) \cup (\{p\})$. Note that Z is a compact subset of $\mathbb{R}^2$. Let $x_n \in M_n$, for each $n \in \mathbb{N}$, and define $S = \{x_n \mid n \in \mathbb{N}\} \cup \{p\}$. Since $\lim_{i\to\infty} M_i = \{p\}$, S is compact. Let $X = Z/S$. Then for each $A \in 2^X$,

$$\mathcal{T}(A) = \begin{cases} X & \text{if } p \in A; \\ \bigcup_{i=1}^{k} M_{n_i} & \text{if } p \notin A, \text{ and } \{M_{n_1}, \ldots, M_{n_k}\} = \{M_n \mid M_n \cap A \neq \emptyset\}. \end{cases}$$

Thus, $|\mathcal{T}(2^X)| = |\{\bigcup_{i\in F} M_i \mid i \in F, F \text{ is finite}\}|$. Therefore, $\mathcal{T}(2^X)$ is countably infinite.

5. Products

We consider the set function $\mathcal{T}$ defined on the product of two continua X and Y. We start considering the family of $\mathcal{T}$-closed sets. Then we show that the continuity of $\mathcal{T}$ on $\mathcal{C}(X \times Y)$ implies the continuity of $\mathcal{T}$, giving a partial positive answer to [8, Question 9.2.10]. We also prove that the compactness of $\mathcal{T}(2^{X\times Y})$ implies the local connectedness of $X \times Y$.

Notation 5.1. If X is a continuum, then

- $\mathfrak{T}(X) = \{A \in 2^X : \mathcal{T}(A) = A\}$;
  and
- $\mathfrak{T}_{\mathcal{C}}(X) = \mathfrak{T}(X) \cap \mathcal{C}(X)$.

The set $\mathfrak{T}(X)$ is known as the collection of $\mathcal{T}$-closed subsets of X (see [1]).

Lemma 5.2. *Let $Z = X \times Y$, where X and Y are continua. Suppose that we have one of the following:*

1. *$\mathfrak{T}(Z)$ is compact;*
   or
2. *$\mathfrak{T}_{\mathcal{C}}(Z)$ is compact.*

Then Z is locally connected.

Proof. Assuming that $\mathcal{H} \in \{\mathfrak{T}(Z), \mathfrak{T}_{\mathcal{C}}(Z)\}$ is compact, we show that X and Y are locally connected. To see that X is locally connected, it suffices to show that $\mathcal{T}(A) = A$ for each $A \in \mathcal{C}(X)$ (see [8, Theorem 3.1.32]). Let $A \in \mathcal{C}(X)$. It is clear that $\mathcal{T}(X) = X$. Hence, we assume that $A \neq X$. Let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of proper subcontinua of X such that $\lim_{n\to\infty} B_n = Y$. Note that $\mathcal{T}(A \times B_n) = A \times B_n$ for each $n \in \mathbb{N}$, [8, Corollary 3.2.11]. Thus, $A \times B_n \in \mathcal{H}$ for each $n \in \mathbb{N}$. By the compactness of $\mathcal{H}$, we have $A \times Y \in \mathcal{H}$ and $\mathcal{T}(A \times Y) = A \times Y$.

Let $x \notin A$. Let y_0 be an arbitrary point of Y. Since $\mathcal{T}(A \times Y) = A \times Y$, $(x, y_0) \notin \mathcal{T}(A \times Y)$. Hence, there exists a continuum W such that $(x, y_0) \in \mathrm{Int}_Z(W)$ and $W \cap (A \times Y) = \emptyset$. Since $\pi_1 \colon Z \to X$ is open, it is clear that $x \in \mathrm{Int}_X(\pi_1(W))$, $\pi_1(W)$ is a continuum, and $\pi_1(W) \cap A = \emptyset$. Thus, $x \notin \mathcal{T}(A)$ and $\mathcal{T}(A) \subseteq A$. Therefore, $\mathcal{T}(A) = A$ and X is locally connected. Similarly we prove that Y is locally connected. Therefore, Z is locally connected. $\square$

The following result extends [7, Corollary 4.2] and gives a partial answer to [8, Question 9.2.10].

Theorem 5.3. *Let X and Y be continua, and let $Z = X \times Y$. Then the following are equivalent:*

1. *$\mathcal{T}$ is continuous;*
2. *$\mathcal{T}|_{\mathcal{C}(Z)}$ is continuous;*
3. *$\mathfrak{T}(Z)$ is a continuum;*
4. *$\mathfrak{T}(Z)$ is compact;*
5. *$\mathfrak{T}_{\mathcal{C}}(Z)$ is a continuum;*
6. *$\mathfrak{T}_{\mathcal{C}}(Z)$ is compact;*
7. *Z is locally connected.*

Proof. It is clear from the definitions that 1 implies 2, 3 implies 4, and 5 implies 6. If $\mathcal{T}$ is continuous, then $\mathcal{T}$ is idempotent, [8, Theorem 3.3.7]. Hence, $\mathcal{T}(2^X) = \mathfrak{T}(Z)$ and $\mathfrak{T}(Z)$ is a continuum. Thus, 1 implies 3. Similarly, we may show that 1 implies 5.

We see that 2 implies 6. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{T}_{\mathcal{C}}(Z)$ such that $\lim_{n\to\infty} A_n = A$, for some $A \in \mathcal{C}(Z)$. Since $\mathcal{T}|_{\mathcal{C}(Z)}$ is continuous, $\lim_{n\to\infty} \mathcal{T}(A_n) = \mathcal{T}(A)$. But $\mathcal{T}(A_n) = A_n$ for each $n \in \mathbb{N}$. Thus, $\mathcal{T}(A) = A$ and $A \in \mathfrak{T}_{\mathcal{C}}(Z)$. Therefore, $\mathfrak{T}_{\mathcal{C}}(Z)$ is compact.

Note that 4 implies 7 and 6 implies 7, follow from Lemma 5.2. Finally, if Z is locally connected, $\mathcal{T}$ is the identity, [8, Theorem 3.1.31]. Therefore, $\mathcal{T}$ is continuous and 7 implies 1. □

Corollary 5.4. *Let X and Y be continua, and let $Z = X \times Y$. If $\mathcal{T}|_{\mathcal{C}(Z)}$ is continuous, then $\mathcal{T}^2(A) = \mathcal{T}(A)$ for each $A \in \mathcal{C}(Z)$.*

Proof. Since $\mathcal{T}|_{\mathcal{C}(Z)}$ is continuous, $\mathcal{T}$ is continuous, by Theorem 5.3. By [8, Theorem 3.3.7], $\mathcal{T}$ is idempotent. In particular, $\mathcal{T}$ is idempotent on continua. □

Lemma 5.5. *Let X and Y be continua, and let $Z = X \times Y$. If $\mathcal{T}(2^Z)$ is compact, then $\mathcal{T}$ is surjective.*

Proof. Observe that $\mathcal{F}_\infty(Z) \subseteq \mathcal{T}(2^Z)$, [8, Theorem 3.7.17]. Since $\mathcal{F}_\infty(Z)$ is dense in 2^Z, $\mathcal{T}(2^Z) = 2^Z$. Therefore, $\mathcal{T}$ is surjective. □

Lemma 5.6. *Let $Z = X \times Y$, where X and Y are continua. If $\mathcal{T}(2^Z)$ is closed, then $X \times \{y_0\} \in \mathfrak{T}_{\mathcal{C}}(Z)$, for each $y_0 \in Y$.*

Proof. Let $y_0 \in Y$. Observe that $\mathcal{T}(D) = D$, for each proper closed $D \subsetneq X \times \{y_0\}$, [8, Corollary 3.2.11]. Since $\mathcal{T}(2^Z)$ is closed, $X \times \{y_0\} \in \mathcal{T}(2^Z)$. Hence, there exists a closed subset E of $X \times \{y_0\}$ such that $\mathcal{T}(E) = X \times \{y_0\}$. It is clear that $E = X \times \{y_0\}$ [8, Corollary 3.2.11]. Therefore, $X \times \{y_0\} \in \mathfrak{T}_{\mathcal{C}}(Z)$. □

The following result gives a partial answer to [2, Question 3.10].

Theorem 5.7. *Let $Z = X \times Y$, where X and Y are continua. If $\mathcal{T}(2^Z)$ is compact, then Z is locally connected.*

Proof. We see that both X and Y are locally connected. We prove that X is locally connected. The proof for Y is similar. Let $A \in \mathcal{C}(X)$. By [8, Theorem 3.1.32], it is enough to prove that $\mathcal{T}(A) = A$.

Observe that $\mathcal{T}(X) = X$. Hence, we may suppose that $X \setminus A \neq \emptyset$. Note that there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X \setminus A$ such that $\mathrm{Cl}_X(\{x_n\}_{n\in\mathbb{N}}) = \{x_n\}_{n\in\mathbb{N}} \cup \mathrm{Bd}_X(A)$. Let $B = \{x_n\}_{n\in\mathbb{N}} \cup \mathrm{Bd}_X(A)$. Since $\mathcal{T}_Z$ is surjective (Lemma 5.5), there exists $L \in 2^Z$ such that $\mathcal{T}_Z(L) = B \times Y$. Observe that $\{x_k\} \times Y$ is a component of $B \times Y$, for each $k \in \mathbb{N}$. Hence, $\mathcal{T}_Z(L \cap (\{x_k\} \times Y)) = \{x_k\} \times Y$ for every $k \in \mathbb{N}$ [8, Corollary 3.1.75]. We know that $\mathcal{T}_Z(R) = R$ for each closed proper subset R of $\{x_k\} \times Y$ [8, Theorem 3.7.8]. Thus, $L \cap (\{x_k\} \times Y) = \{x_k\} \times Y$ and $\{x_k\} \times Y \subseteq L$. Since x_k is an arbitrary point, $\bigcup_{i \in \mathbb{N}}(\{x_i\} \times Y) \subseteq L$. Therefore, $B \times Y \subseteq L$ and $\mathcal{T}_Z(B \times Y) = B \times Y$.

Since the natural projection onto the first factor, $\pi\colon Z \to X$, is open and monotone, we have that $\mathcal{T}_X(B) = B$ [8, Theorem 3.1.80 (e)].

Now, we see that $\mathcal{T}_X(A) = A$. Let $x \in X \setminus A$. We consider two cases:

Case 1. $x \in X \setminus B$.

Then there exists a subcontinuum S of X such that $x \in \mathrm{Int}_X(S)$ and $S \cap B = \emptyset$. Since $x \notin A$ and $S \cap \mathrm{Bd}_X(A) = \emptyset$, we have that $S \cap A = \emptyset$. Thus, $x \in X \setminus \mathcal{T}_X(A)$.

Case 2. $x = x_k$, for some $k \in \mathbb{N}$.

Let $B_0 = B \setminus \{x_k\}$. The same argument that is used above shows that $\mathcal{T}_X(B_0) = B_0$. Thus, $x \in X \setminus \mathcal{T}(A)$ by the Case 1.

Therefore, X is locally connected. □

By Theorems 5.3 and 5.7, we have the following theorem.

Theorem 5.8. *Let X and Y be continua. Let $Z = X \times Y$. The following are equivalent:*

1. *$\mathcal{T}$ is continuous;*
2. *$\mathcal{T}(2^Z)$ is a continuum;*
3. *$\mathcal{T}(2^Z)$ is compact;*
4. *Z is locally connected.*

Question 5.9. Let $Z = X \times Y$ where X and Y are continua. If $\mathcal{T}_Z(\mathcal{C}(Z))$ is compact, then does it follow that Z is locally connected?

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The hyperspace of $\mathcal{T}$-closed subcontinua

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Abstract

The concept of $\mathcal{T}$-closed set of a continuum was defined and studyed by D. P. Bellamy et al. in [1], where $\mathcal{T}$ denotes the Jones's set function. Given a continuum X, in this paper we define the hyperspace $C_{\mathcal{T}}(X)$ as the collection of all $\mathcal{T}$-closed subcontinua of X. We present examples of continua X for which this hyperspace has n elements for each positive integer n or it is the closure of the harmonic sequence or it is the Cantor set. We study the connectedness, compactness and density of $C_{\mathcal{T}}(X)$. Finally, we relate the continuity of $\mathcal{T}$ with the structure of this hyperspace.

Keywords: Continuum, Hyperspace, $\mathcal{T}$-closed subcontinua, $\mathcal{T}$-closed set.
2000 MSC: Primary: 54B20, Secondary: 54B05, 54F15.

1. Introduction

A continuum is a nonempty compact connected metric space. Given a continuum X, by a hyperspace of X, we mean a specified collection of subsets of X endowed with the Hausdorff metric [7, Definition 2.1 and Theorem 2.2].

Given a continuum X, throughout this paper 2^X, $C(X)$ and $F_1(X)$ will denote *the hyperspace of closed and nonempty subsets* of X, *the hyperspace of subcontinua* of X and *the hyperspace of one-point set of* X, respectively. These hyperspaces have been widely studied in the literature; in particular, it is well known that 2^X and $C(X)$ are arcwise connected [7, Corollary 14.9] and

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$F_1(X)$ is an isometric copy of X [13, Exercise 0.63.4]. The reader interested in these and others hyperspaces can consult [7] and [13].

In [3], Prof. F. B. Jones defined the set function $\mathcal{T} : P(X) \to P(X)$ for a compact metric space X and for each $A \in P(X)$ by

$$\mathcal{T}(A) = \{x \in X : \text{for each subcontinuum } W \text{ of } X \text{ such that } x \in \text{int}_X(W), \\ \text{we have that } W \cap A \neq \emptyset\}$$

This function is known as the *Jones's set function* $\mathcal{T}$. Many properties related to these function have been studied, in particular there exist characterizations of some classes of continua using this concept, for example it is known that X is locally connected if and only if $\mathcal{T}(A) = A$ for every $A \in 2^X$ [11, Theorem 3.1.31] and X is an indecomposable continuum if and only if $\mathcal{T}(\{p\}) = X$ for each $p \in X$ [11, Theorem 3.1.34]. For more properties of the set function $\mathcal{T}$ the reader is referred to [3] and [11].

In [1], D. P. Bellamy et al. defined the concept of being $\mathcal{T}$-closed subset as follows: a subset A of a continuum X is said to be *$\mathcal{T}$-closed subset* of X provided that $\mathcal{T}(A) = A$. In particular, in [1, Theorems 4.1 and 4.10], we find characterizations of these kind of elements, also it is showed that the monotone preimage of a $\mathcal{T}$-closed set is $\mathcal{T}$-closed [1, Theorem 3.16].

In this paper, we define the *hyperspace of $\mathcal{T}$-closed subcontinua* of a continuum X as follows:

$$C_{\mathcal{T}}(X) = \{A \in C(X) : A \text{ is } \mathcal{T}\text{-closed}\}.$$

In order to study this hyperspace, after preliminaries, this paper is organized as follows:

In Section 3, we present some examples for which $C_{\mathcal{T}}(X)$ is either or compact but not connected or arcwise connected but not compact, in particular, we construct a continuum X that satisfies that $C_{\mathcal{T}}(X)$ is homeomorphic to the Cantor set (Example 3.13 and Theorem 3.17). Section 4 is divided in two parts; in the first one, we analyzed the connectedness of $C_{\mathcal{T}}(X)$ for the product of continua as a consequence of this we show that if X is the cone of a continuum, then $C_{\mathcal{T}}(X)$ is contractible (Corollary 4.8), in the second part we focus our attention on density of $C_{\mathcal{T}}(X)$ as a subspace of $C(X)$ for some compactifications of the ray, in particular we give a characterization of continua belonging to the $Class(W)$ (Theorem 4.15). In Section 5, the compactness of $C_{\mathcal{T}}(X)$ is analyzed; we show that, if X is a nonlocally connected smooth fan, then $C_{\mathcal{T}}(X)$ is not compact (Proposition 5.4) and we conjecture that does not exist a nonlocally connected fan for which $C_{\mathcal{T}}(X)$ is compact. Finally, in Section 6, we use the continuity of $\mathcal{T}$ to study the hyperspace $C_{\mathcal{T}}(X)$, in particular we show that if $\tilde{X}$ is a X-Lewis continuum (this kind of spaces are defined here), then $2^{\tilde{X}}_{\mathcal{T}}$ is homeomorphic to $2^{X}_{\mathcal{T}}$ and $C_{\mathcal{T}}(\tilde{X})$ is homeomorphic to $C_{\mathcal{T}}(X)$ (Corollary 6.5).

2. Preliminaries

Given a subset A of a continuum X with metric d, the closure, the boundary and the interior of A are denoted by $\mathrm{cl}_X(A)$, $\mathrm{bd}_X(A)$ and $\mathrm{int}_X(A)$, respectively. We omit the subindex if there is no confusion.

Given a finite collection of nonempty subsets $U_1, \ldots, U_m$ of X, we define $\langle U_1, \ldots, U_m \rangle$ as:

$$\left\{ A \in 2^X : A \subset \bigcup_{j=1}^{m} U_j \textit{ and } A \cap U_j \neq \emptyset \textit{ for each } j \in \{1, \ldots, m\} \right\}$$

It is well known that the family of all subset of 2^X of the form $\langle U_1, \ldots, U_m \rangle$, where $m > 0$ and $U_1, \ldots, U_m$ are nonempty open subsets of X, defines a basis for a topology for 2^X called the Vietoris topology [7, Definition 1.1 and Theorem 1.2], and that the Vietoris topology is equivalent to the topology induced by the Hausdorff metric [13, Theorem 0.13].

Given a continuum X and a subset A of X, it is not dificult to see that the Jones's set function $\mathcal{T}$ can be described in the following way:

$$\mathcal{T}(A) = X - \{x \in X : \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in \mathrm{int}(W) \subset W \subset X - A\}.$$

Throughout this paper we will use this equivalence.

Let X be a continuum, we define the hyperspaces $C_{\mathcal{T}}(X)$ and $2^X_{\mathcal{T}}$ as:

$$C_{\mathcal{T}}(X) = \{A \in C(X) : \mathcal{T}(A) = A\},$$

and

$$2^X_{\mathcal{T}} = \{A \in 2^X : \mathcal{T}(A) = A\}.$$

These hyperspace are endowed with the topology as subspaces of 2^X.

We say that the Jones's set function $\mathcal{T}$ is continuous for a continuum X provided that the function $\mathcal{T} : 2^X \to 2^X$ is continuous, and we say that $\mathcal{T}$ is *idempotent on* X provided that $\mathcal{T}^2(A) = \mathcal{T}(A)$ for each $A \in 2^X$.

A *map* means a continuous function and a *mapping* is a surjective map. A mapping between continua $f : X \to Y$ is *weakly confluent* provided that for each subcontinuum K of Y, there exists a component $\mathcal{C}$ of $f^{-1}(K)$ such that $f(\mathcal{C}) = K$. A continuum X is said to be in the *Class*(W), written $X \in$ *Class*(W), provided that for any continuum Y and any mapping $f : Y \to X$, f is weakly confluent. A mapping between continua $f : X \to Y$ is said to be *atomic*, if $Q = f^{-1}(f(Q))$ for each subcontinuum Q of X such that $f(Q)$ is nondegenerated.

A continuum is *decomposable* if it is the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A subcontinuum Y of a continuum X is said to be *terminal* provided that for any subcontinuum K of X such that $K \cap Y \neq \emptyset$, we have that either $K \subset Y$ or $Y \subset K$.

A continnum X is *connected im kleinen at* a point x in X (*cik* at x) if for each closed subset F of X such that $F \subset X - \{x\}$, then there exists a subcontinuum W of X such that $x \in \mathrm{int}(W) \subset W \subset X - F$ (compare with [12, Definition 5.10]).

We say that X is *aposyndetic at* p *with respect to* q provided that there exists a subcontinuum W of X such that $p \in \mathrm{int}(W) \subset W \subset X - \{q\}$. X is *aposyndetic at* p if X is aposyndetic at p with respect to each point of $X - \{p\}$. We say that X is *aposyndetic* provided that X is aposyndetic at each of its points.

A *compactification of the ray* is a compact space X containing a topological copy of $(0, 1]$ as a dense subset. The complement of $(0, 1]$ in X is called the *remainder* of the compactification.

A *dendroid* is an arcwise connected continuum such that the intersection of any two of its subcontinua is connected. A *fan* is a dendroid with exactly one ramification point v, which will be called the vertex of the fan. The set of end point of a continuum X is going to be denoted by $E(X)$. If $n \geq 3$, a simple n-od is a fan with $|E(X)| = n$.

To finish this section, we present some known results and properties of $C_{\mathcal{T}}(X)$ that can be obtained easily from general theory of $\mathcal{T}$.

Lemma 2.1. *[4, Lemma 4.3] Let* X *be a continuum. If* U *is an open set* *such that X is locally connected at x, for each $x \in U$, then $\mathcal{T}(A) = A$, for each $A \in 2^X$ and $A \subset U$.*

Theorem 2.2. *[5, Teorema 3.11] Let X be a continuum. Then $\mathcal{T}$ is continuous if and only if there exist a locally connected continuum Y and a monotone open mapping $f : X \to Y$ such that $\mathcal{T}_X(A) = f^{-1}(f(A))$, for each $A \in 2^X$.*

The following results follow immediately from [11, Theorem 3.1.28], [11, Theorem 3.1.32] and [11, Theorem 3.1.34], respectively.

Theorem 2.3. *A continuum X is aposyndetic if and only if $F_1(X) \subset C_{\mathcal{T}}(X)$.*

Theorem 2.4. *A continuum X is locally connected if and only if $C_{\mathcal{T}}(X) = C(X)$.*

Theorem 2.5. *If X is an indecomposable continuum, then $C_{\mathcal{T}}(X) = \{X\}$.*

Note that if $X = X_1 \cup X_2$ is a continuum, where X_1 and X_2 are indecomposable continua such that $X_1 \cap X_2 = \{p\}$, then $C_{\mathcal{T}}(X) = \{X\}$ and X is decomposable. This shows that the converse of the Theorem 2.5 is not true.

3. Examples

In this section we present some interesting examples which will help to understand this hyperspace from a geometrical point of view, also these examples will help us in future sections and motivate problems that remain open.

The following space is an example of a dendroid for which $C_{\mathcal{T}}(X)$ is not connected and noncompact.

Example 3.1. In the Euclidean plane $\mathbb{R}^2$, let L be the convex segment from $(-1, 0)$ to $(0, 0)$ and for each $n \in \mathbb{N}$, let L_n be the convex segment from $(-1, 0)$ to $(0, \frac{1}{n})$. *The harmonic fan* is the continuum defined by

$$X = \left(\bigcup_{n=1}^{\infty} L_n \right) \cup L.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence contained in $L_1 - \{(-1, 0)\}$, such that $\lim x_n = (-1, 0)$. Note that $\mathcal{T}(\{x_n\}) = \{x_n\}$ for each $n \in \mathbb{N}$ and $\mathcal{T}(\{(-1, 0)\}) = L$. This shows that $C_{\mathcal{T}}(X)$ is not compact.

Now, observe that $V = \langle L_1 - \{(-1,0)\}\rangle \cap C_{\mathcal{T}}(X)$ and $W = \langle X, X - L_1\rangle \cap C_{\mathcal{T}}(X)$ are open, disjoint and nonempty subsets in $C_{\mathcal{T}}(X)$ such that $V \cup W = C_{\mathcal{T}}(X)$. Hence $C_{\mathcal{T}}(X)$ is not connected.

The following space is an example for which $C_{\mathcal{T}}(X)$ is arcwise connected and noncompact.

Example 3.2. Using the notation of Example 3.1, let $L' = \{(x,y) \in \mathbb{R}^2 : (-x,y) \in L\}$ and for each $n \in \mathbb{N}$, let $L'_n = \{(x,y) \in \mathbb{R}^2 : (-x,y) \in L_n\}$. Set $M = L \cup L'$ and $M_n = L_n \cup L'_n$ for each $n \in \mathbb{N}$. The *harmonic suspension* is the continuum defined by

$$X = \left[\bigcup_{n=1}^{\infty} M_n\right] \cup M.$$

The vertices of the harmonic suspension are $(-1,0)$ and $(1,0)$.

Affirmation 3.3. *If X is the harmonic suspension, then $C_{\mathcal{T}}(X) = \{B \in C(X) : M \subset B\} \cup \{B \in C(X) : (-1,0) \notin B$ or $(1,0) \notin B\}$.*

Proof. [$\subseteq$] Let $A \in C_{\mathcal{T}}(X)$ and suppose that $M \not\subset A$. If $(-1,0), (1,0) \in A$, then there exists a component C of $X - A$ contained in M. Note that $\mathrm{int}(C) = \emptyset$, which is a contradiction to [1, Theorem 4.3]. Hence $(-1,0) \notin A$ or $(1,0) \notin A$.

[$\supseteq$] Let $A \in \{B \in C(X) : M \subset B\} \cup \{B \in C(X) : (-1,0) \notin B$ or $(1,0) \notin B\}$. We consider two cases.

Case 1. $M \subset A$.

If $x \in X - A$, then $x \in M_n - \{(-1,0),(1,0)\}$ for some $n \in \mathbb{N}$. Since X is locally connected in x, there exists a subcontinuum B of X such that $x \in \mathrm{int}(B)$ and $B \cap A = \emptyset$. Therefore we have that $T(A) = A$.

Case 2. $(-1,0) \notin A$ or $(1,0) \notin A$.

Without loss of generality, suppose that $(-1,0) \notin A$. Since X has the property of Kelley and $X - A$ is open and arcwise connected, by [1, Theorem 4.10], we have that $\mathcal{T}(A) = A$.

This finishes the proof of this affirmation. □

Affirmation 3.4. *If X is the harmonic suspension, then $C_{\mathcal{T}}(X)$ is arcwise connected.*

Proof. Let $A \in C_{\mathcal{T}}(X)$. By Affirmation 3.3, it is enough to consider two cases.

Case 1. $M \subset A$.

By Theorem [11, Theorem 1.8.20], there exists an order arc $\alpha : [0,1] \to C(X)$ from A to X. Since $M \subset \alpha(t)$ for every $t \in [0,1]$, then $\alpha(t) \in C_{\mathcal{T}}(X)$ for every $t \in [0,1]$.

Case 2. $(-1,0) \notin A$ or $(1,0) \notin A$.

Without loss of generality, suppose that $(-1,0) \notin A$. In this case, it is not difficult to see that there exists an order arc $\alpha : [0,1] \to C(X)$ from A to X such that $(-1,0) \in \alpha(t)$ if and only if $t = 1$. Hence, by Affirmation 3.3, we have that $\alpha(t) \in C_{\mathcal{T}}(X)$ for every $t \in [0,1]$.

From Cases 1 and 2, we infer that $C_{\mathcal{T}}(X)$ is arcwise connected and we finish the proof of this affirmation. □

Affirmation 3.5. *If X is the harmonic suspension, then $C_{\mathcal{T}}(X)$ is not compact.*

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of arcs such that $A_n \subset M_1$, $(-1,0) \notin A_n$ for every $n \in \mathbb{N}$ and $\lim A_n = M_1$. By Affirmation 3.3, we have that $\mathcal{T}(A_n) = A_n$ for every $n \in \mathbb{N}$. Since $\mathcal{T}(M_1) = M_1 \cup M$, we conclude that $C_{\mathcal{T}}(X)$ is not compact. □

The following space is an example for which $C_{\mathcal{T}}(X)$ is compact and non-connected and the structure of it will be important in future constructions.

Example 3.6. In the Euclidean plane $\mathbb{R}^2$, let $\{X_n\}_{n=1}^{\infty}$ be a sequence of indecomposable continua contained in $[0,1] \times [-1,1]$ such that:

(1) $X_n \subset [\frac{n-1}{n}, \frac{n}{n+1}] \times [-1,1]$;

(2) $X_n \cap \mathrm{bd}([\frac{n-1}{n}, \frac{n}{n+1}] \times [-1,1]) = \{(\frac{n-1}{n}, 0), (\frac{n}{n+1}, 0)\}$;

(3) $\lim X_n = \{(1,0)\}$.

We consider the continuum $X = [\bigcup_{n=1}^{\infty} X_n] \cup \{(1,0)\}$.

Note that X satisfies the following properties:

(a) $X_n \cap X_m \neq \emptyset$ if and only if $|m-n| \leq 1$;

(b) $X_n \cap X_{n+1} = \{(\frac{n}{n+1}, 0)\}$;

(c) $X \cap \mathrm{bd}([0,1] \times [-1,1]) = \{(0,0), (1,0)\}$.

Affirmation 3.7. *If X is the continuum of Example 3.6, then $C_{\mathcal{T}}(X) = \{X, \{(1,0)\}\}$ and hence, compact and nonconnected.*

Proof. [$\supseteq$] To prove this contention, it is enough to show that $\{(1,0)\} \in C_{\mathcal{T}}(X)$. If $(a,b) \in X - \{(1,0)\}$, then $a < 1$ and there exists $m \in \mathbb{N}$ such that $a < \frac{m}{m+1}$. Note that $K = \{(x,y) \in X : x \leq \frac{m}{m+1}\}$ is a subcontinuum of X such that $(a,b) \in \text{int}(K)$ and $(1,0) \notin K$. This shows that $\mathcal{T}(\{(1,0)\}) = \{(1,0)\}$. Thus $\{X, \{(1,0)\}\} \subset C_{\mathcal{T}}(X)$.

[$\subseteq$] Note that if $A \in C(X) - \{X, \{(1,0)\}\}$, then there exists $m \in \mathbb{N}$ such that $X_m \cap A \neq \emptyset$ and $X_m \not\subset A$. Let $(a,b) \in X_m - A$. Since X_m is an indecomposable subcontinuum of X and $\text{int}_X(X_m) = X_m - \{(\frac{m-1}{m}, 0), (\frac{m}{m+1}, 0)\}$, we have that if K is a subcontinuum of X such that $(a,b) \in \text{int}_X(K)$, then $X_m \subset K$, and so $K \cap A \neq \emptyset$. Therefore $(a,b) \in \mathcal{T}(A)$ and we obtain that $A \in C(X) - C_{\mathcal{T}}(X)$. This shows that $C_{\mathcal{T}}(X) \subset \{X, \{(1,0)\}\}$ and we finish the proof of this affirmation. □

Given $m \geq 2$, the following space is an example for which $C_{\mathcal{T}}(X)$ has exactly m elements.

Example 3.8. Let $X = [\bigcup_{n=1}^{\infty} X_n] \cup \{(1,0)\}$ be the space defined in Example 3.6 and $m \geq 2$. In the Euclidean space $\mathbb{R}^3$, let $Y_n = \{(x, y, \frac{x}{n}) \in \mathbb{R}^3 : (x,y) \in X\}$ for every $n \in \mathbb{N}$, we defined the continua Y^m by

$$Y^m = \bigcup_{i=1}^{m-1} Y_i.$$

Similar ideas as in Affirmation 3.7 can be used to show that $C_{\mathcal{T}}(Y^m) = \{\{(1, 0, \frac{1}{n})\} : n = 1, \ldots, m-1\} \cup \{Y^m\}$, and hence, $|C_{\mathcal{T}}(Y^m)| = m$. As a consequence, we have the following proposition.

Proposition 3.9. *If $m \in \mathbb{N}$, then there exists a continuum X such that $|C_{\mathcal{T}}(X)| = m$.*

The following space is an example for which $C_{\mathcal{T}}(X)$ is homeomorphic to the closure of the harmonic sequence.

Example 3.10. Let X be the continuum defined in Example 3.6 and for every $n \in \mathbb{N}$, let $Y_n = \{(x, y, \frac{x}{n}) \in \mathbb{R}^3 : (x,y) \in X\}$. We consider the continuum Y defined by

$$Y = [\bigcup_{n=1}^{\infty} Y_n] \cup \{(x,y,0) \in \mathbb{R}^3 : (x,y) \in X\}.$$

Proposition 3.11. *Let* Y *be the continuum defined in Example 3.10,* $C_{\mathcal{T}}(Y) = \{\{(1,0,\frac{1}{n})\} : n \in \mathbb{N}\} \cup \{(1,0,0,)\} \cup \{Y\}$.

Proof. [$\supseteq$] Let $n \in \mathbb{N}$ and let $(x,y,z) \in Y - \{(1,0,\frac{1}{n})\}$. We consider two cases:

Case 1. $x < 1$.

In this case, there exists $m \in \mathbb{N}$ such that $x < \frac{m}{m+1}$. Note that $K = \{(x,y,z) \in Y : x \leq \frac{m}{m+1}\}$ is a subcontinuum of Y such that $(x,y,z) \in \text{int}(K)$ and $(1,0,\frac{1}{n}) \notin K$, thus $(x,y,z) \notin \mathcal{T}(\{(1,0,\frac{1}{n})\})$.

Case 2. $x = 1$.

In this case $K = Y - [Y_n - \{(0,0,0)\}]$ is a subcontinuum of Y such that $(x,y,z) \in \text{int}(K)$ and $(1,0,\frac{1}{n}) \notin K$, thus $(x,y,z) \notin \mathcal{T}(\{(1,0,\frac{1}{n})\})$.

From Cases 1 and 2 we obtain that $\{(1,0,\frac{1}{n}) \in \mathbb{R}^3 : n \in \mathbb{N}\} \subset C_{\mathcal{T}}(Y)$.

Let $(x,y,z) \in Y - \{(1,0,0)\}$. We consider two cases:

Case 1. $x < 1$.

There exists $m \in \mathbb{N}$ such that $x < \frac{m}{m+1}$. Note that $K = \{(x,y,z) \in Y : x \leq \frac{m}{m+1}\}$ is a subcontinuum of Y such that $(x,y,z) \in \text{int}(K)$ and $(1,0,0) \notin K$, thus $(x,y,z) \notin \mathcal{T}(\{(1,0,0)\})$.

Case 2. $x = 1$.

Since $x = 1$, there exists $m \in \mathbb{N}$ such that $z = \frac{1}{m}$. Then $K = \bigcup_{n=1}^{m} Y_n$ is a subcontinuum of Y such that $(1,0,\frac{1}{m}) \in \text{int}(K)$ and $(1,0,0) \notin K$. Thus $(1,0,\frac{1}{m}) \notin \mathcal{T}(\{(1,0,0)\})$.

From Cases 1 and 2 we obtain that $\{(1,0,0)\} \in C_{\mathcal{T}}(Y)$.

Finally, since $Y \in C_{\mathcal{T}}(Y)$, we conclude that $\{(1,0,\frac{1}{n}) \in \mathbb{R}^3 : n \in \mathbb{N}\} \cup \{(1,0,0)\} \cup \{Y\} \subset C_{\mathcal{T}}(Y)$.

[$\subseteq$] Let $A \in C(Y) - [\{(1,0,\frac{1}{n}) \in \mathbb{R}^3 : n \in \mathbb{N}\} \cup \{(1,0,0)\} \cup \{Y\}]$. We will denoted by X_m^n the corresponding copy of X_m contained in Y_n.

Note that there exist $m,n \in \mathbb{N}$, such that, $X_m^n \cap A \neq \emptyset$ and $X_m^n \not\subset A$, then let $(x,y,z) \in X_m^n - A$ and we consider two cases.

Case 1. $z \neq 0$.

Similar ideas as in Example 3.6 can be used to show that, if K is a subcontinuum of Y such that $(x,y,z) \in \text{int}(K)$, then $X_m^n \subset K$ and so $K \cap A \neq \emptyset$. This shows that $(x,y,z) \in \mathcal{T}(A)$, and we obtain that $A \notin C_{\mathcal{T}}(Y)$.

Case 2. $z = 0$.

Note that, if K is a subcotinuum of Y such that $(x, y, 0) \in \text{int}(K)$, then there exists $r \in \mathbb{N}$ such that $X_m^s \subset K$ for every $r < s$. Since $\lim\limits_{s\to\infty} X_m^s \times \{s\} = X_m \times \{0\}$, we obtain that $X_m \times \{0\} \subset K$. This shows that $(x, y, 0) \in \mathcal{T}(A)$, and thus $A \notin C_{\mathcal{T}}(Y)$.

From Cases 1 and 2, we obtain that $A \in C(X) - C_{\mathcal{T}}(X)$. This shows that $C_{\mathcal{T}}(Y) \subset \{(1, 0, \frac{1}{n}) \in \mathbb{R}^3 : n \in \mathbb{N}\} \cup \{(1, 0, 0)\} \cup \{Y\}$ and we finish the proof of this proposition. □

We give some tools to construct a continuum X such that $C_{\mathcal{T}}(X)$ is homeomorphic to the Cantor set.

Let X be the continuum defined in Example 3.6. We define the *Caterpillar Continuum* as:

$$\mathcal{K} = X \cup \{(-x, y) \in \mathbb{R}^2 : (x, y) \in X\}.$$

Let $\pi : \mathcal{K} \to [-1, 1]$ defined by $\pi((x, y)) = x$. By (1), (2) and (3) of Example 3.6, we have the following:

(i) π is a continuous function;

(ii) $\pi^{-1}([\frac{n-1}{n}, \frac{n}{n+1}]) = X_n$ for every $n \in \mathbb{N}$;

(iii) $\pi^{-1}([-\frac{n}{n+1}, -\frac{n-1}{n}]) = \{(-x, y) \in \mathbb{R}^2 : (x, y) \in X_n\}$ for every $n \in \mathbb{N}$;

(iv) $\mathcal{K}$ is contained in $[-1, 1] \times [-1, 1]$;

(v) $\mathcal{K} \cap \text{bd}([-1, 1] \times [-1, 1]) = \{(-1, 0), (1, 0)\}$.

Similar ideas as in Affirmation 3.7 can be used to show the following.

Proposition 3.12. $C_{\mathcal{T}}(\mathcal{K}) = \{\{(-1, 0)\}, \{(1, 0)\}, \mathcal{K}\}$.

Given $a, b \in \mathbb{R}$ with $a < b$, we consider $f_a^b : \mathcal{K} \to \mathbb{R}^2$ defined by $f_a^b((x, y)) = \frac{b-a}{2}(x, y) + (\frac{a+b}{2}, 0)$. Note that f_a^b is a homeomorphism that satisfies the following properties:

(1) $f_a^b((-1, 0)) = (a, 0)$, $f_a^b((0, 0)) = (\frac{a+b}{2}, 0)$ and $f_a^b((1, 0)) = (b, 0)$;

(2) $f_a^b(\mathcal{K}) \subset [a, b] \times [-\frac{b-a}{2}, \frac{b-a}{2}]$;

(3) $f_a^b(\mathcal{K}) \cap bd([a, b] \times [-\frac{b-a}{2}, \frac{b-a}{2}]) = \{(a, 0), (b, 0)\}$.

Example 3.13. Let $\mathcal{C}$ be the Cantor middle-third set contained in $[0,1]$ (see [12, Definition 7.5]). In the Euclidean plane $\mathbb{R}^2$, we consider the continuum:

$$X_{\mathcal{C}} = (\mathcal{C} \times \{0\}) \cup \left[\bigcup \{f_a^b(\mathcal{K}) : a, b \in \mathcal{C}, a < b \text{ and } [a,b] \cap \mathcal{C} = \{a,b\}\}\right].$$

Geometrically, the continuum $X_{\mathcal{C}}$ is the result of add a copy of the Caterpillar continuum in the middle third interval deleted in the construction of the Cantor middle-third set.

Let $\pi : X_{\mathcal{C}} \to [0,1]$ be defined by $\pi((x,y)) = x$. Note that π is a continuous function such that $\pi^{-1}(c) = \{(c,0)\}$ for every $c \in \mathcal{C}$ and $(c,0)$ is a cut point of $X_{\mathcal{C}}$ if and only if $c \in \mathcal{C} - \{0,1\}$.

Proposition 3.14. *$C_{\mathcal{T}}(X_{\mathcal{C}}) = \{\pi^{-1}([c_1,c_2]) \in C(X_{\mathcal{C}}) : c_1, c_2 \in \mathcal{C}$ and $c_1 \leq c_2\}$.*

Proof. $[\supseteq]$ Let $c_1, c_2 \in \mathcal{C}$ be such that $c_1 \leq c_2$ and let $(x,y) \in X_{\mathcal{C}} - \pi^{-1}([c_1,c_2])$. We may assume, without loss of generality, that $x < c_1$. Note that there exists $n \in \mathbb{N}$ and $c_3 \in \mathcal{C}$ such that $c_3 < c_1$, $[c_3, c_1] \cap \mathcal{C} = \{c_3, c_1\}$ and $x < \pi(f_{c_3}^{c_1}(\frac{n}{n+1}, 0))$.

Therefore, $A = \pi^{-1}([0, \pi(f_{c_3}^{c_1}((\frac{n}{n+1}, 0)))])$ is a subcontinuum of $X_{\mathcal{C}}$ having (x,y) in its interior such that $A \cap \pi^{-1}([c_1,c_2]) = \emptyset$. This shows that $(x,y) \notin \mathcal{T}(\pi^{-1}([c_1,c_2]))$ and we conclude this contention.

$[\subseteq]$ Let $A \in C_{\mathcal{T}}(X_{\mathcal{C}})$ and let $a, b \in [0,1]$ be such that $a \leq b$ and $\pi(A) = [a,b]$.

Claim 1. $\pi^{-1}(t) \subset A$ for every $t \in [a,b]$.

Suppose that there exists $t \in [a,b]$ such that $\pi^{-1}(t) \not\subset A$, since $(c,0)$ is a cut point of $X_{\mathcal{C}}$, we have that $t \notin \mathcal{C}$. Let $c_1, c_2 \in \mathcal{C}$ be such that $[c_1,c_2] \cap \mathcal{C} = \{c_1, c_2\}$ and $c_1 < t < c_2$. By construction of the Caterpillar continuum and $X_{\mathcal{C}}$, we may assume, without loss of generality, that there exists $n \in \mathbb{N}$ such that $\pi^{-1}(t) \subset f_{c_1}^{c_2}(X_n)$. Thus, we infer that $f_{c_1}^{c_2}(X_n) \subset \mathcal{T}(A) = A$, which is a contradiction. This finishes the proof of Claim 1.

Now suppose that $b \notin \mathcal{C}$, then $b < 1$ and there exist $c_1, c_2 \in \mathcal{C}$ such that $c_1 < b < c_2$ and $[c_1,c_2] \cap \mathcal{C} = \{c_1,c_2\}$. Without loss of generality by construction of the Caterpillar continuum and $X_{\mathcal{C}}$, there exists $n \in \mathbb{N}$ such that $f_{c_1}^{c_2}(X_n) \cap A \neq \emptyset$ and $f_{c_1}^{c_2}(X_n) \not\subset A$. Hence, we infer that $f_{c_1}^{c_2}(X_n) \subset \mathcal{T}(A) = A$, a contradiction. This shows that $b \in \mathcal{C}$. Similar ideas show that $a \in \mathcal{C}$. This finishes the proof of this contention. □

Proposition 3.15. *$C_{\mathcal{T}}(X_{\mathcal{C}})$ is a perfect set.*

Proof. This follows immediately from the fact that $\mathcal{C}$ is a perfect set and Proposition 3.14. $\square$

Proposition 3.16. *$C_{\mathcal{T}}(X_{\mathcal{C}})$ is totally disconnected.*

Proof. Let $A, B \in C_{\mathcal{T}}(X_{\mathcal{C}})$. By Proposition 3.14, there exist $c_1, c_2, c_3, c_4 \in \mathcal{C}$, such that $A = \pi^{-1}([c_1, c_2])$ and $B = \pi^{-1}([c_3, c_4])$. We may assume, without loss of generality, that $c_2 < c_4$ and there exits $c \in \mathcal{C}$ such that $c_2 < c \leq c_4$ and $[c_2, c] \cap \mathcal{C} = \{c_2, c\}$. Note that $\mathcal{V} = \langle \pi^{-1}([0, \pi(f^c_{c_2}(0)))) \rangle \cap C_{\mathcal{T}}(X_{\mathcal{C}})$ and $\mathcal{W} = \langle X_{\mathcal{C}}, X_{\mathcal{C}} - \pi^{-1}([0, \pi(f^c_{c_2}(0))) \rangle \cap C_{\mathcal{T}}(X_{\mathcal{C}})$ are two open and disjoint subsets in $C_{\mathcal{T}}(X_{\mathcal{C}})$, such that $A \in \mathcal{V}$, $B \in \mathcal{W}$ and $C_{\mathcal{T}}(X_{\mathcal{C}}) = V \cup W$. This shows that $C_{\mathcal{T}}(X_{\mathcal{C}})$ is totally disconnected. $\square$

From Propositions 3.15, 3.16 and [11, Theorem 2.2.9], we obtain the following result.

Theorem 3.17. *$C_{\mathcal{T}}(X_{\mathcal{C}})$ is homemorphic to the Cantor set.*

To finish this section we present the following general problem.

Problem 3.18. *Given a compact and metric space Y, does there exist a continuum X such that $C_{\mathcal{T}}(X)$ is homeomorphic to Y?*

In according with the above problem, we conjecture that there is no continuum X such that $C_{\mathcal{T}}(X)$ is an arc.

4. Connectedness and density of $C_{\mathcal{T}}(X)$

In this section, we analyze the connectedness and the density of the hyperspace $C_{\mathcal{T}}(X)$. By Theorem 2.4, we focus in nonlocally connected continua. In order to study the connectedness of $C_{\mathcal{T}}(X)$, we can see that there exist dendroids for wich $C_{\mathcal{T}}(X)$ is nonconnected (see Example 3.1) and there exist arcwise connected and nonlocally connected continua such that $C_{\mathcal{T}}(X)$ is arcwise connected (see Example 3.2). Continue with this matter, we analyze the connectedness in the family of the compactifications of the ray. For this purpose the following results will be useful throughout this section.

Recall that, if X is a compactification of the ray $(0, 1]$ with remainder Z, then Z is a terminal subcontinuum of X and the following equality holds:

$$C(X) = C((0, 1]) \cup C(Z) \cup \{A \in C(X) : A \text{ is homeomorphic to } X\}. \quad (4.1)$$

Lemma 4.1. *If X is a compactification of the ray $(0,1]$ with remainder Z, then*

$$C_{\mathcal{T}}(X) = C((0,1]) \cup \{Z\} \cup \{A \in C(X) : A \textit{ is homeomorphic to } X\}.$$

Proof. [$\supseteq$] By Lemma 2.1, we have that $C((0,1]) \subset C_{\mathcal{T}}(X)$. Let $A \in C(X)$ be such that A is homeomorphic to X and let $x \in X - A$. Since X is locally connected at x, there exists a connected open subset U of X such that $x \in U$ and $\mathrm{cl}(U) \cap A = \emptyset$. Since $\mathrm{cl}(U)$ is a subcontinuum of X, we obtain that $A \in C_{\mathcal{T}}(X)$. Similar ideas show that $Z \in C_{\mathcal{T}}(X)$.

[$\subseteq$] By the equality in 4.1, it is enough to show that if A is a proper subcontinuum of Z, then $A \notin C_{\mathcal{T}}(X)$. Let $A \in C_{\mathcal{T}}(X)$ and suppose that A is a proper subcontinuum of Z. Since Z is a terminal subcontinuum of X, we have that: If K is a subcontinuum of X such that $x \in \mathrm{int}(K)$, then $Z \subset K$. This shows that $A \notin C_{\mathcal{T}}(X)$. □

Lemma 4.2. *If X is a compactification of the ray $(0,1]$ with remainder Z, then $C_{\mathcal{T}}(X) \subset cl(C((0,1]))$*

Proof. By Lemma 4.1, we have that $C_{\mathcal{T}}(X) = C((0,1]) \cup \{Z\} \cup \{A \in C(X) : A \text{ is homeomorphic to } X\}$. Hence, this result follows from the fact that Z and every subcontinuum homeomophic to X can be aproximated by elements in $C((0,1])$. □

Theorem 4.3. *If X is a compactification of the ray $(0,1]$ with remainder Z, then $C_{\mathcal{T}}(X)$ is connected.*

Proof. By Lemmas 4.1 and 4.2, we have that $C((0,1]) \subset C_{\mathcal{T}}(X) \subset \mathrm{cl}(C((0,1]))$. From [7, Example 5.1], it is not difficult to see that $C((0,1])$ is connected. Hence, $C_{\mathcal{T}}(X)$ is connected. □

The following results will help us to study the connectedness of $C_{\mathcal{T}}(X)$ when X is the product of some continua.

Lemma 4.4. *Let X and Y be continua and $A \in 2^X$. If Y is aposyndetic, then $\mathcal{T}(A \times \{y\}) = A \times \{y\}$ for every $y \in Y$.*

Proof. Let $A \in 2^X$ and $y \in Y$. We shall prove that $\mathcal{T}(A \times \{y\}) \subset A \times \{y\}$. For this purpose, let $(a,b) \in [X \times Y] - [A \times \{y\}]$. We consider two cases.

Case 1. $b \neq y$.

Since Y is aposyndetic, there exists a subcontinuum K of Y such that $b \in \text{int}(K) \subset Y - \{y\}$. Therefore, $X \times K$ is a subcontinuum of $X \times Y$ such that $(a, b) \in \text{int}(X \times K) \subset X \times K \subset [X \times Y] - [A \times \{y\}]$. Thus $(a, b) \notin \mathcal{T}(A \times \{y\})$.

Case 2. $b = y$.

In this case, we have that $a \notin A$. Let $c \in Y - \{y\}$ and let F be a closed subset of X such that $a \in \text{int}(F)$ and $F \cap A = \emptyset$. Then $K = [F \times Y] \cup [X \times \{c\}]$ is a subcontinuum of $X \times Y$ such that $(a, b) \in \text{int}(K) \subset K \subset [X \times Y] - [A \times \{y\}]$. Thus $(a, b) \notin \mathcal{T}(A \times \{y\})$.

From Cases 1 and 2, we obtain that $\mathcal{T}(A \times \{y\}) = A \times \{y\}$ and we finish the proof of this lemma. □

Theorem 4.5. *If X is a continuum, then $C_{\mathcal{T}}(X \times [0, 1])$ is connected.*

Proof. By [11, Corollary, 3.3.9] and Theorem 2.3, we have that $F_1(X \times [0, 1]) \subset C_{\mathcal{T}}(X \times [0, 1])$. Let $A \in C_{\mathcal{T}}(X \times [0, 1]) - F_1(X \times [0, 1])$, if π_2 denotes the natural projection from $X \times [0, 1]$ to $[0, 1]$, then there exist $a, b \in [0, 1]$ such that $a \leq b$ and $\pi_2(A) = [a, b]$. We consider two cases:

Case 1. $a = b$.

In this case $A \subset X \times \{a\}$. Let $(x, a) \in A$, by [7, Theorem 14.6], there exists an order arc $\alpha : [0, 1] \to C(X \times [0, 1])$ such that $\alpha(0) = \{(x, a)\}$ and $\alpha(1) = A$. Note that for every $t \in [0, 1]$, we have that $\alpha(t) \subset A \subset X \times \{a\}$. Since $[0, 1]$ is aposyndetic, by Lemma 4.4, we have that $\alpha([0, 1]) \subset C_{\mathcal{T}}(X \times [0, 1])$.

Case 2. $a < b$.

Let $H : (X \times [a, b]) \times [0, 1] \to X \times [a, b]$ be defined by $H((x, s), t) = (x, s(1 - t) + at)$. Note that H is a homotopy such that $H((x, s), 1) = (x, a)$ and $H((x, s), 0) = (x, s)$ for every $(x, s) \in X \times [0, 1]$ and $H(X \times [a, b], t)$ is homeomorphic to $X \times [a, b]$ for each $t \in [0, 1)$. Thus, $H(A, t)$ is homeomorphic to A for each $t \in [0, 1)$ and we have that $H(A, t) \in C_{\mathcal{T}}(X \times [0, 1])$ for every $t \in [0, 1)$. Note that $H(A, 1)$ is a subcontinuum of $X \times \{a\}$. Let $(x, a) \in H(A, 1)$, by [7, Theorem 14.6], there exists an order arc $\alpha : [0, 1] \to C(X \times [0, 1])$ such that $\alpha(0) = \{(x, a)\}$ and $\alpha(1) = H(A, 1)$, as in Case 1, we have $\alpha([0, 1]) \subset C_{\mathcal{T}}(X \times [0, 1])$. We define $\beta : [0, 1] \to C(X)$ by:

$$\beta(t) = \begin{cases} H(A, 2t) & \text{if } t \in [0, 1/2] \\ \alpha(2 - 2t) & \text{if } t \in [1/2, 1] \end{cases}$$

Note that β is a map such that $\beta([0, 1]) \subset C_{\mathcal{T}}(X \times [0, 1])$ and $A, \{(x, y)\} \in \beta([0, 1])$.

Case 1 and 2 show that for every $A \in C_{\mathcal{T}}(X)$, there exists a map $\gamma : [0,1] \to C(X \times [0,1])$ such that $A \in \gamma([0,1])$, $\gamma([0,1]) \subset C_{\mathcal{T}}((X \times [0,1]))$ and $\gamma([0,1]) \cap F_1(X \times [0,1]) \neq \emptyset$. Hence, $C_{\mathcal{T}}((X \times [0,1]))$ is connected. □

Corollary 4.6. *If X is a continuum, then $C_{\mathcal{T}}(X \times [0,1])$ is arcwise connected.*

Proof. Since $C_{\mathcal{T}}(X \times [0,1])$ is a continuum, by Cases 1 and 2 of Theorem 4.5, we obtain that for every element $A \in C_{\mathcal{T}}(X \times [0,1])$, there exist $(x,t) \in X \times [0,1]$ and an embedding $\gamma : [0,1] \to C_{\mathcal{T}}(X \times [0,1])$ such that $\gamma(0) = \{(x,t)\}$ and $\gamma(1) = A$. Hence, we only need to show that any two elements in $F_1(X \times [0,1])$ belong to an arc in $C_{\mathcal{T}}(X \times [0,1])$.

Let $(x_1,y_1), (x_2,y_2) \in F_1(X \times [0,1])$. By [7, Theorem 14.6], for $i = 1,2$, there exists an order arc $\alpha_i : [0,1] \to C(X \times [0,1])$ such that $\alpha_i(0) = \{(x_i,y_i)\}$ and $\alpha_i(1) = X \times \{y_i\}$. Now, note that $\gamma : [0,1] \to C(X \times [0,1])$ defined by $\gamma(t) = X \times \{y_1(1-t) + y_2 t\}$ is a map such that $\gamma(0) = X \times \{y_1\}$ and $\gamma(1) = X \times \{y_2\}$. By Lemma 4.5, we have that $\alpha_1(t), \alpha_2(t), \gamma(t) \in C_{\mathcal{T}}(X \times [0,1])$ for every $t \in [0,1]$. Therefore, $\Gamma = \alpha_1([0,1]) \cup \gamma([0,1]) \cup \alpha_2([0,1])$ is a path from $\{(x_1,y_1)\}$ to $\{(x_2,y_2)\}$. Since $C_{\mathcal{T}}(X \times [0,1])$ is Hausdorff there exists an arc from (x_1,y_1) to (x_2,y_2) contained in $C_{\mathcal{T}}(X \times [0,1])$. □

Similar ideas as in Theorem 4.5 and Corollary 4.6 can be extended to show the following result.

Corollary 4.7. *Let X be a continuum. If T_n is a simple n-od, then $C_{\mathcal{T}}(X \times T_n)$ is arcwise connected.*

For the following result, recall that for a metric space X, $Cone(X)$ denotes the topological cone of X, which is defined as the quotient space $(X \times [0,1])/(X \times \{0\})$.

Theorem 4.8. *If X is a continuum, then $C_{\mathcal{T}}(Cone(X))$ is contractible.*

Proof. Let $q : X \times [0,1] \to Cone(X)$ be the quotient mapping. If $v_0 = q(X \times \{0\})$, then it is easy to see that $\{v_0\} \in C_{\mathcal{T}}(Cone(X))$. Let $H : (X \times [0,1]) \times [0,1] \to X \times [0,1]$ be the function defined by $H((x,t),s) = (x, t(1-s))$. Note that H is a mapping fulfilling $H((x,t),0) = (x,t)$ and $H((x,t),1) = (x,0)$, even more, if $A \subset X \times (0,1]$, then $H(A,s)$ is homeomorphic to A for every $s \in [0,1)$. Let $G : Cone(X) \times [0,1] \to Cone(X)$ be the function defined by $G(A,s) = q(H(q^{-1}(A) \times \{s\}))$. It is easy to show that G is a well defined mapping satisfying $G(A,0) = A$ and $G(A,1) = v_0$.

Finally, it is clear that if $A \in C_{\mathcal{T}}(Cone(X))$, then $G(A, s)$ is homeomorphic to A for every $s \in [0,1)$, since $G(A,1) = \{v_0\}$ for every $A \in C_{\mathcal{T}}(Cone(X))$, we obtain that $G|_{C_{\mathcal{T}}(Cone(X))} : C_{\mathcal{T}}(Cone(X)) \times [0,1] \to C_{\mathcal{T}}(Cone(X))$ is a contraction from $C_{\mathcal{T}}(Cone(X))$ to v_0. □

Since every contractible space is arcwise connected, we obtain the following corollary.

Corollary 4.9. *If X is a continuum, then $C_{\mathcal{T}}(Cone(X))$ is arcwise connected.*

To finish the first part of this section, we present a result related with the continuity of $\mathcal{T}$ and the connectedness of $C_{\mathcal{T}}(X)$ (general result about the continuity of $\mathcal{T}$ and $C_{\mathcal{T}}(X)$ will be presented in Section 6). Also we give some problems related with this topic.

Theorem 4.10. *Let X be a continuum. If $\mathcal{T}$ is continuous in X, then $\mathcal{T} : C(X) \to C_{\mathcal{T}}(X)$ is well defined continuous and surjective function and hence, $C_{\mathcal{T}}(X)$ is connected.*

Proof. We only need to show that $\mathcal{T} : C(X) \to C_{\mathcal{T}}(X)$ is well defined and surjective. Since $\mathcal{T}$ is continuous, by [11, Theorem 3.28], we have that $\mathcal{T}^2 = \mathcal{T}$. Therefore, $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$ for every $A \in C(X)$ and we obtain that $\mathcal{T}$ is well defined. Now, if $A \in C_{\mathcal{T}}(X)$, then $\mathcal{T}(A)$ is an element in $C(X)$ such that $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A) = A$. Therefore $\mathcal{T}$ is surjective. □

Problem 4.11. *Let X be a continuum and let Y be a finite graph. Is $C_{\mathcal{T}}(X \times Y)$ connected (arcwise connected)?*

Problem 4.12. *Let X be a compact metric space. If $Sus(X)$ denotes the suspension of X, is $C_{\mathcal{T}}(Sus(X))$ connected (arcwise connected)?*

Problem 4.13. *Let X and Y be a continua. If Y is contractible and locally connected, is $C_{\mathcal{T}}(X \times Y)$ connected (arcwise connected)?*

Note that the above problems are particular cases of the following general problem.

Problem 4.14. *Let X be a continuum. If X is aposyndetic, is $C_{\mathcal{T}}(X)$ connected (arcwise connected)?*

Now, we present some results related with the density of $C_{\mathcal{T}}(X)$ as a subspace of $C(X)$. For this purpose, we focus our atention in continua belonging to the $Class(W)$.

Theorem 4.15. *Let X be a continuum. Then, $X \in Class(W)$ if and only if every compactification Y of the ray $(0,1]$ with remainder X, satisfies that cl$(C_{\mathcal{T}}(Y)) = C(Y)$.*

Proof. Suppose that $X \in Class(W)$ and let Y be a compactification of the ray $(0,1]$ with remainder X. By [7, (c) of Theorem 67.1], we have that $C(Y)$ is a compactification of $C((0,1])$. Hence, by Lemma 4.1, we obtain that $C(Y) = \mathrm{cl}(C(((0,1])) \subset \mathrm{cl}(C_{\mathcal{T}}(Y)) \subset C(Y)$. This shows that $\mathrm{cl}(C_{\mathcal{T}}(Y)) = C(Y)$.

On the other hand, suppose that Y is a compactification of the ray $(0,1]$ with remainder X, then by hipothesis $\mathrm{cl}(C_{\mathcal{T}}(Y)) = C(Y)$. By Lemma 4.2, we have that $C_{\mathcal{T}}(Y) \subset \mathrm{cl}(C((0,1]))$, which implies that $\mathrm{cl}(C((0,1])) = C(Y)$. Hence, by [7, (d) of Theorem 67.1] we obtain that $X \in Class(W)$. □

Corollary 4.16. *If X is a compactification of the ray with remander a chainable continuum, then cl$(C_{\mathcal{T}}(X)) = C(X)$.*

Proof. By [14, Theorem 4], we have that every chainable continuum belongs to $Class(W)$. Hence, the result follows from Theorem 4.15. □

Corollary 4.17. *Let Y be an arcwise connected continuum. Then, Y is an arc if and only if every compactification X of the ray with remainder Y, satisfies that cl$(C_{\mathcal{T}}(X)) = C(X)$.*

Proof. Suppose that every compactification X of the ray $(0,1]$ with remainder Y satisfies that $\mathrm{cl}(C_{\mathcal{T}}(X)) = C(X)$. By Theorem 4.15, we have that $Y \in Class(W)$, since Y is arcwise connected, from [7, Theorem 67.1, (a)] and [13, Theorem 14.73.20], we have that Y is an arc.

The other implication follows from Corollary 4.16. □

5. Compactness of $C_{\mathcal{T}}(X)$

In this section we analize the compactness of $C_{\mathcal{T}}(X)$. We recall that there exist dendroids for which $C_{\mathcal{T}}(X)$ is not compact (Example 3.1); also by Lemma 4.1, it is easy to verify that if X is a compactification of the ray, then $C_{\mathcal{T}}(X)$ is not compact. However, we have continua X for which

$C_{\mathcal{T}}(X)$ either has n elements (Proposition 3.9) or it is the harmonic sequence (Example 3.10) or it is the Cantor set (Theorem 3.17). We will study the compactness in the class of n-indecomposable continua and in the class of smooth fans.

Recall that if X is a continuum and there exist n continua whose union is X such that no one of them is a subset of the union of the others, then X is said to be the *finished sum* of these n continua. If the continuum X is the finished sum of n continua and it is not the finished sum of $n+1$ continua, then X is said to be *n-indecomposable*. From the definition, we have that every 1-indecomposable continuum is indecomposable.

Theorem 5.1. *Let $n \geq 1$. If X is an n-indecomposable continuum, then $C_{\mathcal{T}}(X) = \{X\}$ and hence, compact.*

Proof. Let X be an n-indecomposable continuum. By [15, Theorem 2], we have that $X = \bigcup_{i=1}^{n} K_i$ where K_i is an indecomposable continuum for each $i \in \{1, \ldots, n\}$.

Let $A \in C(X) - \{X\}$, then there exists $m \in \{1, \ldots, n\}$, such that $K_m \cap A \neq \emptyset$ and $K_m \not\subset A$. By [2, Theorem 7], we have that if $x \in K_m \cap A$, then $K_m \subset \mathcal{T}(\{x\})$. Hence, $K_m \subset \mathcal{T}(A)$ and thus $A \notin C_{\mathcal{T}}(X)$. This shows that $C_{\mathcal{T}}(X) = \{X\}$. □

Note that the space X defined in the Example 3.6, is not n-indecomposable continuum for every $n \in \mathbb{N}$ but $C_{\mathcal{T}}(X)$ is compact. This says that the converse of Theorem 5.1 is not true.

Theorem 5.2. *Let X be a continuum. If $\mathcal{T}$ is continuous in X, then $C_{\mathcal{T}}(X)$ is compact.*

Proof. By Theorem 4.10, we have that $\mathcal{T} : C(X) \to C_{\mathcal{T}}(X)$ is a well defined mapping. Hence, $C_{\mathcal{T}}(X)$ is compact. □

Now, we focus our attention in fans. Given a and b two points of a dendroid X, then ab will denote the arc contained in X with end points a and b. If $a = b$, then $ab = \{a\}$. For a continuum X, the set of all the points in X such that X is not cik at x will be denoted by N_X.

Proposition 5.3. *If X is a nonlocally connected fan with vertex v, then $\mathcal{T}(\{v\}) = \{v\} \cup N_X$.*

Proof. [$\subseteq$] Let $x \in \mathcal{T}(\{v\}) - \{v\}$ and suppose that X is cik at x. Then there exists a subcontinuum A of X such that $x \in \text{int}(A)$ and $v \notin A$, which is a contradiction. This shows that $x \in N_X$ and we have that $\mathcal{T}(\{v\}) \subset \{v\} \cup N_X$.

[$\supseteq$] Let $x \in N_X$. By [12, Theorem 5.12], there exist a subcontinuum K and a sequence of mutually disjoint subcontinua $\{K_n\}_{n=1}^{\infty}$ of X such that $K \subset N_X$, $x \in K$, $K_n \cap K = \emptyset$ for every $n \in \mathbb{N}$ and $\lim K_n = K$. Note that for every $n \in \mathbb{N}$, if $a \in K_n$, then $v \in xa$. Now, if W is a subcontinuum of X such that $x \in \text{int}(W)$, then there exists $n \in \mathbb{N}$ such that $W \cap K_n \neq \emptyset$, since W is arcwise connected, we obtain that $v \in W$. Therefore, $x \in \mathcal{T}(\{v\})$ and we conclude that $\{v\} \cup N_X \subset \mathcal{T}(\{v\})$. □

Let X be a smooth fan with vertex v. By [6, Proposition 4], we have that X can be embedded into the cone over the Cantor set. Thus, it is easy to show that: if $x \in X - \{v\}$, then there exists a unique end point e of X such that $v \notin xe$ and $\mathcal{T}(xe) = xe$.

Proposition 5.4. *If X is a nonlocally connected smooth fan, then $C_{\mathcal{T}}(X)$ is not compact.*

Proof. Let X be a nonlocally connected smooth fan with vertex v and let $x, y \in X - \{v\}$ be such that $x \in N_X$ and $y \notin \mathcal{C}_x$, where $\mathcal{C}_x$ is the component of $X - \{v\}$ having x. Denote by e the end point of X such that $v \notin ye$ and let $\{y_n\}_{n=1}^{\infty}$ be a sequence in $ve - \{v\}$ such that $\lim y_n = v$. From the fact that $\mathcal{T}(y_n e) = y_n e$ for every $n \in \mathbb{N}$, we have that $\lim \mathcal{T}(y_n e) = ve$. On the other hand, by Proposition 5.3, we have that $\mathcal{T}(ve) = \{v\} \cup N_X$ and since $x \in N_X - ve$, we obtain that $\mathcal{T}(ve) \neq ve$. This shows that $C_{\mathcal{T}}(X)$ is not compact. □

Proposition 5.5. *Let X be a nonlocally connected fan with vertex v. Then $\mathcal{T}(\{v\}) = X$ if and only if $\{X\}$ is a isolated point of $C_{\mathcal{T}}(X)$.*

Proof. Suppose that $\mathcal{T}(\{v\}) = X$. Let a_1 and a_2 be two end points in X, note that $W = \langle X, va_1 - v, va_2 - v \rangle \cap C_{\mathcal{T}}(X)$ is an open subset in $C_{\mathcal{T}}(X)$ such that $X \in W$. If $A \in W$, then $A \cap va_1 - v \neq \emptyset$ and $A \cap va_2 - v \neq \emptyset$, which implies that $v \in A$. Hence, by [11, Proposition 3.1.7], we have that $X = \mathcal{T}(\{v\}) \subset \mathcal{T}(A) = A$ and thus $A = X$. This shows that $W = \{X\}$ and we obtain that $\{X\}$ is an isolated point of $C_{\mathcal{T}}(X)$.

Now, suppose that X is an isolated point of $C_{\mathcal{T}}(X)$. If $\mathcal{T}(\{v\}) \neq X$, then there exists $x \in X$ such that $x \notin \mathcal{T}(\{v\})$ and thus, there exists a subcontinuum K of X such that $x \in \text{int}(K)$ and $v \notin K$. Hence, if e is the end point in X such that $v \notin xe$, then $xe - \{x\}$ is an open subset of X (this means tha xe is a free arc in X). It is not difficult to see that: if $\{e_n\}_{n=1}^{\infty}$ is a sequence in $xe - \{e\}$ such that $\lim e_n = e$, then $X_n = X - (e_n e - \{e_n\})$ is a $\mathcal{T}$-closed subcontinuum in X for every $n \in \mathbb{N}$ such that $\lim X_n = X$. This implies that X is not an isolate point of $C_{\mathcal{T}}(X)$, a contradiction. □

Concerning to Proposition 5.5, we conjecture the following.

Conjecture 5.6. *If X is a nonlocally connected fan, then $C_{\mathcal{T}}(X)$ is not compact.*

In general, we believe that the above conjecture must be true for nonlocally connected dendroids.

6. $C_{\mathcal{T}}(X)$ and the continuity of $\mathcal{T}$

If $f : X \to Y$ is a mapping between continua, then $\hat{f} : 2^X \to 2^Y$ given by $\hat{f}(A) = f(A)$ is called the induced mapping from 2^X to 2^Y and, it is well known that $\hat{f}$ is continuous ([11, Theorem 1.8.22]).

Theorem 6.1. *Let X be a continuum. If $\mathcal{T}$ is continuous in X, then there exists a continuum Y such that $C_{\mathcal{T}}(X)$ is homeomorphic to $C(Y)$.*

Proof. By Theorem 2.2, there exist a locally connected continuum Y and a monotone open map $f : X \to Y$ such that $\mathcal{T}_X(A) = f^{-1}(f(A))$, for each $A \in 2^X$. We shall prove that $C_{\mathcal{T}}(X)$ is homeomorphic to $C(Y)$.

Let $\hat{f}^{-1} : C(Y) \to C_{\mathcal{T}}(X)$ defined by $\hat{f}^{-1}(A) = f^{-1}(A)$. Since Y is locally connected and f is a monotone mapping, by [1, Theorem 3.16] and [11, Theorem 1.8.24], we have that $\hat{f}^{-1} : C(Y) \to C_{\mathcal{T}}(X)$ is well defined and continuous. Let $E, F \in C(Y)$ be such that $\hat{f}^{-1}(E) = \hat{f}^{-1}(F)$, since f is surjective, we have that $E = f(f^{-1}(E)) = f(f^{-1}(F)) = F$. This shows that $\hat{f}^{-1}$ is injective. Now, if $A \in C_{\mathcal{T}}(X)$, by Theorem 2.2, we have that $A = f^{-1}(f(A))$ and since $f : X \to Y$ is continuous we obtain that $f(A) \in C(Y)$, hence $\hat{f}^{-1}(f(A)) = A$. This shows that $\hat{f}^{-1}$ is a surjective mapping.

Finally, since $\hat{f}|_{C_{\mathcal{T}}(X)} : C_{\mathcal{T}}(X) \to C(Y)$ is the inverse mapping of $\hat{f}^{-1}$, we obtain that $C_{\mathcal{T}}(X)$ is homeomorphic to $C(Y)$. □

The following result can be consulted in [9, Corollary 3.8].

Theorem 6.2. *Let X and Y be continua, let $f : X \to Y$ be an atomic mapping. Then $A \in 2^X_{\mathcal{T}}$ if and only if there exists $B \in 2^Y_{\mathcal{T}}$ such that $A = f^{-1}(B)$.*

Theorem 6.3. *Let X and Y be continua and let $f : X \to Y$ be an atomic mapping. If f is open, then $2^X_{\mathcal{T}}$ is homeomorphic to $2^Y_{\mathcal{T}}$.*

Proof. Let $\hat{f}^{-1} : 2^Y_{\mathcal{T}} \to 2^X_{\mathcal{T}}$ defined by $\hat{f}^{-1}(A) = f^{-1}(A)$. Since f is open mapping, by Theorem 6.2 and [11, Theorem 1.8.24], we have that $\hat{f}^{-1} : 2^Y_{\mathcal{T}} \to 2^X_{\mathcal{T}}$ is well defined, continuous and is surjective. Let $E, F \in 2^Y_{\mathcal{T}}$ be such that $\hat{f}^{-1}(E) = \hat{f}^{-1}(F)$, since f is surjective, we have that $E = f(f^{-1}(E)) = f(f^{-1}(F)) = F$. This shows that $\hat{f}^{-1}$ is injective.

Finally, since $\hat{f}|_{2^X_{\mathcal{T}}} : 2^X_{\mathcal{T}} \to 2^Y_{\mathcal{T}}$ is the inverse mapping of $\hat{f}^{-1}$, we obtain that $2^X_{\mathcal{T}}$ is homeomorphic to $2^Y_{\mathcal{T}}$. □

In order to mention some applications of previous results in this section, recall that a closed partition $\mathcal{D}$ of a topological space X is said to be a *terminal continuous decomposition* of X provided that every element of $\mathcal{D}$ is a terminal subcontinuum of X and the natural projection $\pi : X \to 2^X$ is continuous. Hence, we introduce the following concept.

Let X and $\tilde{X}$ be two continua. We say that $\tilde{X}$ is an X-Lewis continuum, provided that $\tilde{X}$ has a terminal continuous decomposition into pseudo-arcs with the property that the decomposition space is homeomorphic to X.

By [8], we have that if X is a one-dimensional continuum, then there exists a continuum $\tilde{X}$ such that it is a one-dimensional X-Lewis continuum. Hence, from Theorem 6.1, we have the following result.

Corollary 6.4. *Let X be a locally connected continuum. If $\tilde{X}$ is an X-Lewis continuum, then $C(X)$ is homeomorphic to $C_{\mathcal{T}}(\tilde{X})$.*

And as a consequence of Theorem 6.3, we have:

Corollary 6.5. *Let X be a continuum. If $\tilde{X}$ is an X-Lewis continuum, then:*

(1) $2^{\tilde{X}}_{\mathcal{T}}$ *is homeomorphic to* $2^X_{\mathcal{T}}$*,*

(2) $C_{\mathcal{T}}(\tilde{X})$ *is homeomorphic to* $C_{\mathcal{T}}(X)$*.*

Finally, we pose two problems related with the results in this section.

Problem 6.6. *Let X be a continuum. Does there exist a continuum Y such that $C(X)$ is homeomorphic to $C_{\mathcal{T}}(Y)$?*

Note that Theorem 6.1 gives a partial answer to the above problem.

Problem 6.7. *Let X be a continuum for which $C_{\mathcal{T}}(X)$ is a nondegenerate continuum. Is it true that $\mathcal{T}$ is continuous at X?*

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Discusión general y Conclusiones

La función $\mathcal{T}$ de Jones es una función bastante estudiada dentro de la teoría de continuos, un problema abierto respecto a esta función, es la continuidad de la función, de la cual no se sabe mucho al respecto y uno de los resultado conocidos de continuidad ocupa hipotesis demasiado fuertes, esto fue una de las cosas que nos encontramos durante el estudio de nuetro tema, por ejemplo el artículo "Some aspects related to the Jones' set function $\mathcal{T}$" analizamos el hiperespacio de subconjuntos $\mathcal{T}$-cerrados $(2_{\mathcal{T}}(X))$ y el hiperespacio de subcontinuos $\mathcal{T}$-cerrados $(C_{\mathcal{T}}(X))$, algunos de los resultados los obtuvimos suponiendo que los hiperespacios $2_{\mathcal{T}}(X)$ o $C_{\mathcal{T}}$ son compactos la cual es una hipotesis bastante fuerte y con esto logramos obtener resultados que ligan estos dos hiperespacios, también observamos que la compacidad en los hiperespacios $2_{\mathcal{T}}(X)$ y $C_{\mathcal{T}}$ nos garantiza la continuidad de la función $\mathcal{T}$, con esto obtenemos una respuesta parcial a la pregunta sobre la continuidad de la función $\mathcal{T}$, por otra parte también construimos un continuo tal que $\mathcal{T}$ es idempotente en continuos pero no es idempotente en conjuntos cerrados, estos resultados junto con otros fueron incluidos en el artículo.

Siguiendo con nuesto estudio de los hiperespacios $2_{\mathcal{T}}(X)$ y $C_{\mathcal{T}}(X)$, encontramos que trabajar con el hiperespacio de los subconjuntos cerrados y sus imágenes bajo la función $\mathcal{T}$, nos proporciona bastante información la cual nos fue difícil analizar por lo que tratamos de restringir la función $\mathcal{T}$ al hiperespacio de los subcontinuos esperando obtener algunos resultados y obtuvimos algunos resultados pero también en el hiperespacio de continuos nos resulto complicado trabajar con la función $\mathcal{T}$, esto nos motivo a seguir analizando ejemplos de continuos y su hiperespacio de subcontinuos $\mathcal{T}$-cerrados.

Dentro del análisis realizado a los ejemplos de continuos y su hiperespacio de subcontinuos $\mathcal{T}$-cerrados vimos que la compacidad no esta ligada con la conexidad, ni a la densidad, con lo cual nos dimos a la tarea de ver como se comportaba la compacidad y una de las cosas que logramos fue encontrar un continuo tal que su hiperespacio de subcontinuos $\mathcal{T}$-cerrados es homeomorfo al conjunto de Cantor, con esto surgió la siguiente pregunta:

1. Dado un espacio métrico compacto Y, ¿existe un continuo X tal que $C_{\mathcal{T}}(X)$ es homeomorfo a Y?.

Esta pregunta aún sigue abierta y tenemos algunas conjeturas al respecto.

Analizando el hiperespacio de continuos $\mathcal{T}$-cerrados de las compactaciones del rayo, obtuvimos resultados respecto a la densidad de este hiperespacio, dentro de los cuales logramos la caracterización de los continuos de la Clase(W). Para el estudio de conexidad nos volvimos a enfocar en el producto de continuos, en los cuales buscamos la conexidad y logramos probar la arco-conexidad del hiperespacio. En la parte de compacidad, estudiamos los continuos n-indescomponibles los cuales no dan ejemplos sencillos sobre la compacidad del hiperespacio. En abanicos suaves no localmente conexos logramos demostrar que el hiperespacio resulta no compacto.

Durante este trabajo nos dimos cuenta que la conexidad y compacidad fueron temas en los cuales encontramos mayor dificultad y de mayor interés, ya que están muy ligadas a la continuidad de la Funcion $\mathcal{T}$. Los resultados obtenidos al respecto los integramos en el artículo "The hypersapce of $\mathcal{T}$-closed subcontinua".

Conclusiones

El estudio del Hiperespacio de los subcontinuos $\mathcal{T}$-cerrados nos llevó a una de las preguntas mas importantes dentro de la función $\mathcal{T}$: ¿Cuándo es continua la función $\mathcal{T}$? consideramos que esto es un factor que hace un poco complicado el tema. Cabe resaltar que con respecto a la continuidad de $\mathcal{T}$ tenemos una conjetura la cual incluye al hiperespacio de subcontinuos $\mathcal{T}$-cerrados.

Otro tema que también nos resulto de mucho interés y dentro del cual tenemos conjeturas es la aposindesis, la conjetura trata sobre la conexidad del hiperespacio de subcontinuos $\mathcal{T}$-cerrados.

Otro factor a tomar en cuenta al trabajar con la función $\mathcal{T}$ de Jones y el hiperespacio de subcontinuos $\mathcal{T}$-cerrados, es que este tema se debe pensar siempre en continuos no localmente conexos, ya que en los continuos localmente conexos se tiene la continuidad de la función $\mathcal{T}$ y la función resulta ser la identidad y en este caso el hiperespacio de subcontinuos $\mathcal{T}$-cerrado coincide con el hiperespacio de subcontinuos el cual ya fue ampliamente estudiado.

El hiperespacio de subcontinuos $\mathcal{T}$ podría ser una forma alternativa de abordar preguntas sobre la función $\mathcal{T}$ de Jones que aún siguen abiertas.

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