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Continuos cuyo hiperespacio de no estorbadores de sus singulares es un continuo.

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Resumen.

Un continuo es un espacio métrico, conexo, compacto y con más de un punto. El hiperespacio de todos los subconjuntos cerrados y no vacíos de un continuo X dotado de la topología generada por la métrica de Hausdorff es denotado por 2^X. Dado un continuo X, el subsespacio $NB(F_1(X))$ de 2^X consta de todos los los elementos $A \in 2^X - X$ tal que para cada $x \in X - A$, la unión de todos los subcontinuos de X que contienen a x y están contenidos en $X - A$ es un subconjunto denso de X. Los elementos de $NB(F_1(X))$ se denominan subconjuntos no estorbadores de los singulares del continuo X. La investigación realizada durante los estudios de maestría estuvo enfocada a mostrar que para cada espacio métrico Y, existe un continuo X tal que $NB(F_1(X))$ es homeomorfo a Y. Esto responde a una pregunta planteada por J. Camnargo, F. Capulín, E. Castañeda-Alvarado y D. Maya en un artículo en 2019.

Introducción y presentación del objeto de estudio.

Un continuo es un espacio métrico compacto conexo y no degenerado. El hiperespacio de todos los subconjuntos cerrados no vacíos de un continuo X se denota por 2^X, el simbolo $C(X)$ representa el hiperespacio de todos los subcontinuos de X, y $F_1(X)$ denota el hiperespacio que consta de todos los singulares de X. Estos hiperespacios topologizados por la métrica de Hausdorff son continuos. (véase la definición de la métrica de Hausdorff en [**?**, p.11].

Dado un continuo X, $A \in 2^X$ no le estorba a $x \in X - A$ si la union de la familia $\{L \in C(X) : x \in L \subseteq X - A\}$ es un subconjunto denso de X. La colección de todos $A \in 2^X - X$ tal que A no le estorba a cada $x \in X - A$ es denotado por $NB(F_1(X))$ y los elementos de $NB(F_1(X))$ son llamados subconjuntos no estorbadores de $F_1(X)$.

Uno de los temas más atractivos de este hiperespacio es determinar sus propiedades topológicas como subespacio de 2^X, por ejemplo, su compacidad. En primer lugar, [10, Teorema 4.4, Corolario 5.4, p.3617-3618] muestra que el hiperespacio de no estorbadores de un continuo es compacto por ser una curva cerrada simple. Además, [6, p.37-40] exhibe una serie de ejemplos de continuos X para argumentar que $NB(F_1(X))$ puede ser homeomorfo a a cada uno de los siguientes espacios métricos compactos: un conjunto finito, la cerradura de una sucesión convergente, el hiperespacio 2^Z para cada continuo Z, un arco, una 2-celda y el cubo de Hilbert. Por lo que, naturalmente surge la siguiente pregunta

Pregunta 1.1 ¿Para qué espacio métrico compacto Y existe un continuo X tal que $NB(F_1(X))$ es homeomorfo a Y?.

Obsérvese que la pregunta 1.1 es la versión general de [6, Preguntas 4.13, 4.14, 4.15, 4.18, 4.21, p.38-40]. El propósito de este trabajo es responder a la Pregunta 1.1 demostrando que cada espacio métrico compacto Y se puede incrustar en un continuo X de tal manera que $NB(F_1(X)) = F_1(Y)$.

Revisión bibliográfica.

Una gran cantidad de propiedades de un continuo están determinadas por el comportamiento topológico de sus hiperespacios y viceversa. El estudio de estas propiedas es muy amplio [15]. Dentro de esta teoría es muy natural preguntarse si el considerar un subespacio de un hiperespacio cuyos elementos cumplen una característica muy particular se tiene el mismo efecto. Cultivando esta línea de investigación, ha resultado muy interesante estudiar al subespacio que consta de todos los subconjuntos cerrados que no estorban a los singulares. Entre los resultados más importantes sobre este hiperespacio se encuentra la siguiente caracterización(ver Teorema 3.2 de [10]): el único continuo localmente conexo cuyo hiperespacio de subconjuntos cerrados que no estorban a los singulares consta de todos sus singulares, es la curva cerrada simple. Utilizando resultados recientes, en [6], es posible mostrar que tal caracterización es válida para cualqiuier continuo. Cabe mencionar, hasta donde tenemos conocimiento, que la información existente en la literatura sobre este hiperespacio se encuentra vertida únicamente en [4],[6],[9] y [10], de aquí la relevancia de este proyecto.

Planteamiento del problema, objetivos e hipótesis y preguntas de investigación.

Sea X un continuo. Diremos que $B \in 2^X$ es no estorbador de $F_1(X)$ si para cada $x \in X - B$ se cumple que $\bigcup\{G \in C(X) : x \in G \subseteq X - B\}$ es un subconjunto denso de X. Definimos $NB(F_1(X)) = \{B \in 2^X : B$ es no estorbador de $F_1(X)\}$. El hiperespacio 2^X será dotado de la topología generada por la métrica de Hausdorff (ver[15,p.11]). De esta manera 2^X es un continuo y los subsespacios $C(X), F_1(X)$ son subcontinuos de 2^X. Dentro de la teoría, es natural buscar condiciones sobre X bajo las cuales se tiene $NB(F_1(X))$ es un subcontinuo de 2^X. Este problema se ha abordado respondiendo la siguiente pregunta:
Pregunta(Pregunta 4.21 de [4]) ¿ Para qué continuos Y existe un continuo X tal que $NB(F_1(X))$ es homeomorfo a Y?.
Este artículo está dirigido a dar una respuesta más general a la pregunta para el caso de que Y es un espacio métrico compacto.
Objetivos: Ampliar la frontera del comnocimiento mediante la investigación dentro del área de la teoría de continuos y sus hiperespacios, sobre el hiperespacio de los conjuntos cerrados no estorbadores de los singulares de un continuo para seguir cultivando las líneas de generación y aplicación del conocimiento que contribuyen a los avances científicos en esta área de las matemáticas.

Descripción metodológica.

La metología para la realización de este proyecto fue la tradicional en la investigación en matemáticas. Generalmente el proceso de generación del conocimiento en esta área se inicia con el trabajo individual y es seguido de discusiones de las ideas con los profesores a cargo mediante la exposición en diversos seminarios de las demostraciones de los teoremas, los enunciados en los que se sintetizan todos los descubrimientos, mostrando hasta los más finos detalles con la finalidad que la finalidad que la audiencia, en repetidas ocasiones y fallando en cada una de estos intentos, intente vulnerar mediante contraejemplos los argumentos y con esto se pruebe su solidez. Se realizó una revisión bibliográfica para proveer de información relevante la investigación.

Resultados.

All compactum is homeomorphic to the hyperspace of nonblockers of singletons of a continuum

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Abstract

A *continuum* is a nondegenerate compact connected metric space. The hyperspace of all nonempty closed subset of a continuum X topologized by the Hausdorff metric is denoted by 2^X. Given a continuum X, the subspace $\mathcal{NB}(\mathcal{F}_1(X))$ of 2^X consists of all elements $A \in 2^X - \{X\}$ such that for each $x \in X - A$, the union of all subcontinua of X containing x and contained in $X - A$ is a dense subset of X. The members of $\mathcal{NB}(\mathcal{F}_1(X))$ are called *nonblocker subsets of the singletons* of the continuum X. In this paper, we show that each compact metric space Y can be embedded in a continuum X such that the hyperspace of nonblocker subsets of X and Y are homeomorphic. This answers a question posed by J. Camargo, F. Capulín, E. Castañeda-Alvarado and D. Maya.

Keywords: continuum, nonblocker set, semi-boundary

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1. Introduction

A *continuum* is a nondegenerate compact connected metric space. The hyperspace of all nonempty closed subsets of a continuum X is denoted by 2^X, the symbol $\mathcal{C}(X)$ represents the hyperspace of all subcontinua of X, and $\mathcal{F}_1(X)$ denotes the hyperspace consisting of all singletons of X. These hyperspaces topologized by the Hausdorff metric are continua (see the definition of Hausdorff metric in [15, p. 11]).

Given a continuum X, $A \in 2^X$ *does not block* $x \in X - A$ provided that the union of the family $\{L \in \mathcal{C}(X) : x \in L \subseteq X - A\}$ is a dense subset of X. The collection of all $A \in 2^X - \{X\}$ such that

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A does not block each $x \in X - A$ is denoted by $\mathcal{NB}(\mathcal{F}_1(X))$ and the members of $\mathcal{NB}(\mathcal{F}_1(X))$ are called *nonblockers subsets of* $\mathcal{F}_1(X)$.

Several author have studied the notion of nonblocker subsets in hyperspaces (see [2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16]). One of the most attractive topics about this hyperspace is to determine its topological properties as subspace of 2^X, for instance, its compactness. First, [10, Theorem 4.4, Corollary 5.4, pp. 3617, 3618] shows that the hyperspace of nonblocker sets of a continnum can be compact by being a simple closed curve. Next, [4, pp. 37-40] exhibits a serie of examples of continua X to argue that $\mathcal{NB}(\mathcal{F}_1(X))$ can be homeomorphic to each one of the following compact metric spaces: a finite set, the closure of a convergent sequence, the hyperspace 2^Z for each continuum Z, an arc, an 2-cell and the Hilbert cube. So, naturally the question below arises.

Question 1.1. *For which compact metric space Y there exists a continuum X such that $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to Y?*

Observe that Question 1.1 is the general version of [4, Questions 4.13, 4.14, 4.15, 4.18, 4.21, pp. 38-40]. Recently, the answer to this question in [11] (a work done in parallel to this) is all compact metric space.

The purpose of this paper is to answer Question 1.1 by proving that each compact metric space Y can be embedded in a continuum X (topologically distinct to the continuum constructed in [11] despite both works use mainly a continuum consisting of the closure of a sequence of plane indecomposable continua presented in [4, Section 4]) such that $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(Y)$. A highlighted fact is that the notion of semi-boundary in hyperspace is used here in order to simplify and to diversify the proofs.

2. Preliminaries

This section contains the basic notions and previous results necessary to the aim of this paper. Throughout section, the symbol X will be a continuum. The word *mapping* stands for a continuous function.

For a point $p \in X$, the *composant of* p in X is the union of the family consisting of all proper subcontinua of X containing p. A subset κ of X is a *composant* of X provided that there exists $p \in X$ such that κ is the composant of p in X. A continuum is *indecomposable* provided the family of all its composants is an uncountable partition of X (see [17, Theorems 11.13 and 11.17, pp. 202, 204]).

A subset V of X is *continuumwise connected* provided that for each $p, q \in V$, there exists $L \in \mathcal{C}(X)$ such that $p, q \in L \subseteq V$.

For each proper closed subset A of X and $x \in X-A$, define $\kappa(x, A) = \bigcup\{N \in C(X) : x \in N \subseteq X - A\}$.

Proposition 2.1. *If A is a proper closed subset of X and $x \in X - A$, then $\kappa(x, A)$ is a continuumwise connected subset of $X - A$.*

Let

$$\mathcal{NWC}(X) = \left\{ A \in 2^X : \begin{array}{c} \text{the interior of } A \text{ is empty and} \\ X - A \text{ is continuumwise connected} \end{array} \right\}.$$

Given $L, K \in \mathcal{C}(X)$ such that $L \subsetneq K$, an *order arc from L to K* is a mapping $\alpha : [0, 1] \to \mathcal{C}(X)$ such that $\alpha(0) = L$, $\alpha(1) = K$ and $\alpha(s) \subsetneq \alpha(t)$ provided $0 \leq s < t \leq 1$. The existence of order arcs follows from [15, Theorem 14.6, p. 112]. The last fact will be used repeatedly without mentioning explicity. For a point $x \in X$, let $\Lambda(x, X)$ be the collection of all order arc from $\{x\}$ to X.

The next result shows a characterization of the nonblocking property of a proper closed subset of a continuum in terms of the existence of a particular order arc (see [10, Proposition 2.2, p. 3615]).

Proposition 2.2. *Let A be a proper closed subset of X and let $x \in X - A$. Then A does not block x if and only if there exists $\alpha \in \Lambda(x, X)$ such that $\alpha(t) \cap A = \emptyset$ for each $t < 1$.*

Let

$$\mathcal{NB}(\mathcal{F}_1(X)) = \left\{ A \in 2^X : \begin{array}{c} \text{the interior of } A \text{ is empty and} \\ A \text{ does not block each } x \in X - A \end{array} \right\}.$$

The inclusion $\mathcal{NWC}(X) \subseteq \mathcal{NB}(\mathcal{F}_1(X))$ is proved in [9, Theorem 3.2, p. 99].

Let L be a proper subcontinuum of X. The *semi-boundary* $SB(L)$ of $C(L)$ is the set of all $K \in C(L)$ such that there exists a mapping $f : [0, 1] \to \mathcal{C}(X)$ such that $f(0) = K$ and $f(t) - L \neq \emptyset$ for each $t > 0$. This notion is introduced in [12]. By [12, (a) of Theorem 2.1, p. 65], an element $K \in C(L)$ is a member of $SB(L)$ if and only if there exists an order arc α from K to X such that $\alpha(t) - L \neq \emptyset$ for each $t > 0$.

For $L \in \mathcal{C}(X)$ and $x \in L$, let $SB(L, x) = \{K \in SB(L) : x \in K\}$. The next result follows from the definition of semi-boundary in hyperspaces and the existence of order arcs between a pair of comparable subcontinua.

Lemma 2.3. *Let $L \in C(X)$ and let $x \in L$. If $K \in \mathcal{C}(X) - \{X\}$ is such that $x \in K \subseteq L$, then $K \in SB(L, x)$ if and only if there exist $\alpha \in \Lambda(x, X)$ and $s \in [0, 1]$ such that $\alpha(s) = K$ and $\alpha(t) - L \neq \emptyset$ for each $t \in (s, 1]$.*

Lemma 2.4. *Let $L \in \mathcal{C}(X)$ and let $x \in L$. If $A \in \mathcal{NB}(\mathcal{F}_1(X))$ is such that $A \cap K \neq \emptyset$ for each $K \in SB(L, x)$, then $x \in A$.*

Proof. Assume that $x \in X - A$. Choose $\alpha \in \Lambda(x, X)$ such that $\alpha(t) \cap A = \emptyset$ for each $t < 1$. Set $s = \sup\{t \in [0,1] : \alpha(t) \subseteq L\}$ and $T = \alpha(s)$. By Lemma 2.4, T is an element of $SB(L)$. The inclusions $x \in T \subseteq X - A$ contradict the assumption A intersects each element of $SB(L, x)$. Therefore, $x \in A$. $\square$

Let U be a subset of X. Define

$$\begin{aligned} \langle U \rangle &= \left\{A \in 2^X : A \subseteq U\right\} \text{ and} \\ \langle X, U \rangle &= \left\{A \in 2^X : A \cap U \neq \emptyset\right\}. \end{aligned}$$

The family

$$\{\langle U \rangle : U \text{ is an open subset of } X\} \cup \{\langle X, U \rangle : U \text{ is an open subset of } X\}$$

is a subbasis for a topology of 2^X. Such topology is called *Vietoris topology* for 2^X and it coincides with the topology generated by the Hausdorff metric (see [17, Theorem 4.5, p. 54]).

3. Main results

This section is devoted to argue that all compact metric space is the answer of Question 1.1.

The set of all integers is denoted by $\mathbb{Z}$ and the symbol $\mathbb{N}$ represents the set of all positive elements of $\mathbb{Z}$. The space $S = \left\{n^{-1} : n \in \mathbb{N}\right\} \cup \{0\}$ will be endowed with the usual topology of the real line $\mathbb{R}$. Denote each $(x, x) \in \mathbb{R}^2$ by $\overline{x}$.

Let Y be a compact metric space. Use C to denote the ternary Cantor set. Let $f : C \to Y$ be an onto mapping guaranteed by [17, Theorem 7.7, p. 106].

For each $n \in \mathbb{Z}$, let $\varphi_n : \mathbb{R}^2 \to \mathbb{R}^2$ be the homeomorphism defined by

$$\varphi_n((a,b)) = \begin{cases} (-1 + 2^{-n}(1+a), -1 + 2^{-n}(1+b)), & \text{if } n \geq 0, \\ (2 - 2^{n}(1+a), 2 - 2^{n}(1+b)), & \text{if } n < 0. \end{cases}$$

Let Q be an indecomposable plane continuum contained in $[0,1] \times [0,1]$ such that $Q \cap ((\{0,1\} \times [0,1]) \cup ([0,1] \times \{0,1\})) = \{\bar{0}, \bar{1}\}$ and $\bar{1}$ does not belong to the composant of $\bar{0}$ in Q.

For each $n \in \mathbb{Z}$, let $W_n = \varphi_n(Q)$. Define

$$W = \{\overline{-1}, \bar{2}\} \cup \bigcup_{n \in \mathbb{Z}} W_n = \mathrm{Cl}_{\mathbb{R}^2}\left(\bigcup_{n \in \mathbb{Z}} W_n\right).$$

Use similar arguments in the first part of [4, Example 4.9, p. 37] to prove the next result.

Proposition 3.1. *The sets* $\{\overline{-1}\}$ *and* $\{\overline{2}\}$ *are elements of* $\mathcal{NWC}(W)$.

In order to define a usc decomposition of $C \times C \times W \times S$, define

$$\begin{aligned} R &= \left\{(x,y,w,z) \in C \times C \times (W - \{\overline{-1}, \overline{2}\}) \times S : w \neq \overline{0} \text{ or } f(x) \neq f(y)\right\}, \\ E &= \left\{(x,y,\overline{0},z) \in C \times C \times W \times S : f(x) = f(y)\right\}, \text{ and} \\ F(c) &= (f^{-1}(f(c)) \times C \times \{\overline{-1}\} \times S) \cup (C \times f^{-1}(f(c)) \times \{\overline{2}\} \times S) \end{aligned}$$

for each $c \in C$. Set

$$\mathfrak{D} = \{\{(x,y,w,z)\} : (x,y,w,z) \in R\} \cup \{E\} \cup \{F(c) : c \in C\}.$$

Observe that $\mathfrak{D}$ is a partition consisting of closed subsets of $C \times C \times W \times S$. Let $\tau_{\mathfrak{D}}$ be the topology of decomposition for $\mathfrak{D}$ and let $\pi : C \times C \times W \times S \to \mathfrak{D}$ be the natural function.

Theorem 3.2. *The decomposition space* $Z = (\mathfrak{D}, \tau_{\mathfrak{D}})$ *is a continuum.*

Proof. For each convergent sequence $(u_k)_{k\in\mathbb{N}}$ in $C \times C \times W \times S$, the inclusion $\limsup \pi(u_k) \subseteq \pi(\lim u_k)$ holds. Then $(\mathfrak{D}, \tau_{\mathfrak{D}})$ is a usc decomposition of $C \times C \times W \times S$. Invoke [17, Theorem 3.9, p. 40] to conclude that Z is metrizable. Since Z is the continuous image of the compact space $C \times C \times W \times S$ under π, Z is compact. It only remains to prove that Z is connected.

Let $u \in Z$. Choose $(x,y,w,z) \in u$. Set $L_1 = \{x\} \times \{y\} \times W \times \{z\}$ and $L_2 = \{x\} \times \{x\} \times W \times \{z\}$. Then L_1 and L_2 are connected subsets of $C \times C \times W \times S$ such that $u \in \pi(L_1)$, $E \in \pi(L_2)$ and $F(x) \in \pi(L_1) \cap \pi(L_2)$. This implies that $\pi(L_1) \cup \pi(L_2)$ is a connected subset of Z and $u, E \in \pi(L_1) \cup \pi(L_2)$. Thus, each element of Z is contained in a connected subset of Z containing E. Therefore, Z is connected. □

The continuum Z is the first step to find the desired continuum. The next main actor is the set $G = \pi(C \times C \times \{\overline{0}\} \times \{0\})$ and his role will be crucial.

A subspace of a continuum having component uniquely consisting of a one point is called *totally disconnected*. A space such that the family of all its open closed subsets forms a base is called *zero-dimensional*. Both notions coincide in compact metric spaces and the space C possesses them.

Lemma 3.3. *The subspace* G *of* Z *is totally disconnected.*

Proof. Let $u, v \in G$ be different points. We may assume that $u \neq E$. Choose $(x,y,\overline{0},0) \in u$ and $(a,b,\overline{0},0) \in v$. Then $f(x) \neq f(y)$. Let U and V be disjoint open subsets of Y such that $f(x) \in U$ and $f(y) \in V$. Since C is zero-dimensional, there exist open closed subsets L and N of C such that $x \in L \subseteq f^{-1}(U)$, $y \in N \subseteq f^{-1}(V)$ and $(a,b) \in (C \times C) - (L \times N)$. From the fact that π is a closed mapping, it follows that $J=\pi(L\times N\times\{\overline{0}\}\times\{0\})$ and $I=\pi((C\times C\times\{\overline{0}\}\times\{0\})-(L\times N\times\{\overline{0}\}\times\{0\}))$ are closed subsets of G. Observe that $u\in J$, $v\in I$ and $G=J\cup I$. Let us prove that $J\cap I=\emptyset$.

Let $c\in J$. There exists $(s,t)\in L\times N$ such that $\pi(s,t,\overline{0},0)=c$. Then $f(s)\in U$ and $f(t)\in V$. This implies that $f(s)\neq f(t)$. Thus, $c=\pi(s,t,\overline{0},0)=\big\{(s,t,\overline{0},0)\big\}$. Hence, $\pi^{-1}(c)\cap((C\times C\times\big\{\overline{0}\big\}\times\{0\})-(L\times N\times\big\{\overline{0}\big\}\times\{0\}))=\emptyset$, in other words, $c\in G-N$. Therefore, $J\subseteq G-N$.

In conclusion, each subset of G having at least two points is disconnected. So, G is totally disconnected. □

The *pseudo-arc* is the unique hereditarily indecomposable chainable continuum (see [1, Theorem 1, p. 44]).

Let $p\notin Z$ and define $M=G\cup\{p\}$. Endow M with the topology whose base consists of all open subsets of G and of $\{p\}$. Use Lemma 3.3 to prove that M is a totally disconnected compact metric space. Employ the previous fact and [8, Theorem 10, p. 17] together to deduce the result below.

Proposition 3.4. *There exists a pseudo-arc P containing M such that each composant of P contains at most one component of M.*

Set $X=Z\cup P$. Endow X with the topology that results by attaching the closed subset G of both spaces to obtain that X is a continuum. For the rest of the section, both P and Z are considered as subspaces of X.

The next result exhibits relevant topological properties of X.

Proposition 3.5. *The subspace $\{F(c):c\in C\}$ of the continuum X is a topological copy of Y.*

Proof. Define $g:C\to X$ by $g(c)=\pi(c,c,\overline{2},0)=F(c)$ to get a mapping such g is constant in each set $f^{-1}(y)$. By [17, 3.22, p. 45], there exists a mapping $h:Y\to X$ such that $g(c)=h(f(c))$. Observe that h is one-to-one. This proves that h is an embedding and $h(Y)=\{F(c):c\in C\}$. □

The topological copy of Y contained in X is represented by $\hat{Y}$.

Proposition 3.6. *If u is an element of the composant of p in P, then $SB(P,u)=\{P\}$.*

Proof. The inclusion $P\in SB(P,u)$ follows from [13, (c) of Theorem 1.2 p. 65]. Now, let $K\in SB(P,u)$. By Lemma 2.3, there exist $\alpha\in\Lambda(u,X)$ and $s\in[0,1]$ such that $K=\alpha(s)$ and $\alpha(t)-A\neq\emptyset$ for each $t\in(s,1]$. Assume that K is a proper subset of P. This implies that K is contained in the composant u of P. Hence, K and G are disjoint. Since $X-G$ is the union of the disjoint nonempty open subsets $X - P$ and $X - Z$, K must be a subset of $X - Z = P - G$. Now, from the continuity of α, it follows the existence of $\delta > 0$ such that $\alpha([s, s+\delta]) \subseteq \langle X - Z \rangle$. This implies that $\alpha(s+\delta)$ is a subset of $P - G$. It contradicts the choice of s. Therefore, $\alpha(s) = P$. Therefore, $SB(P, u) = \{P\}$. $\square$

Proposition 3.7. *Let $x, y \in C$, let $z \in S$, and let $n \in \mathbb{Z}$. If u is an element of $\pi(\{x\} \times \{y\} \times W_n \times \{z\})$ such that its composant and the set $\pi(\{x\} \times \{y\} \times \varphi_n(\{\overline{0}, \overline{1}\}) \times \{z\})$ are disjoint, then $SB(\pi(\{x\} \times \{y\} \times W_n \times \{z\}), u) = \{\pi(\{x\} \times \{y\} \times W_n \times \{z\})\}$.*

Proof. The proof follows by using a similar argument in the proof of the last result. We repeat it here for completeness. For sake of simplicity, set $J = \pi(\{x\} \times \{y\} \times W_n \times \{z\})$, $v = \pi(x, y, \varphi_n(\overline{0}), z)$ and $q = \pi(x, y, \varphi_n(\overline{1}), z)$. Observe that J is homeomorphic to W_n. So, J is indecomposable and the componsant of v does not contain q.

Invoke [13, (c) of Theorem 1.2 p. 65] to deduce that $J \in SB(J, u)$. Next, let $K \in SB(J, u)$. Lemma 2.3 guarantees the existence of $\alpha \in \Lambda(X, u)$ and $s \in [0, 1]$ such that $K = \alpha(s)$ and $\alpha(t) - J \neq \emptyset$ for each $t \in (s, 1]$. Assume that K is a proper subset of J. Then K must be contained in the composant of u in J. This implies that $K \cap \{v, q\}$ is empty. Let $W^* = \left\{\overline{-1}, \overline{2}\right\} \cup \bigcup_{k \in \mathbb{Z} - \{n\}} W_k$. Observe that W^* and W_n are closed subsets of W such that $W = W_n \cup W^*$. Set $R = \pi(C \times C \times W^* \times S) \cup P$ and $T = \pi(C \times C \times W_n \times S)$. Notice that R and T are closed subset of X satisfying $X = R \cup T$ and $v, q \in R \cap T$. Then K is a subset of the union of the disjoint nonempty open subsets $X - R$ and $X - T$ of X. Since $K - R \neq \emptyset$, the subcontinuum K is contained in $X - R$. By the continuity of α, there exists $\delta > 0$ such that $\alpha([s, s+\delta]) \subseteq \langle X - R \rangle$. Hence, $\alpha(s+\delta) \subseteq T$. Now, J is the component of T containing $\alpha(s)$. Thus, $\alpha(s+\delta)$ is a subset of J. A contradiction. Therefore, $K = J$. $\square$

Proposition 3.8. *Let $x, y \in C$ be such that $f(x) \neq f(y)$ and let $m \in \mathbb{N}$. If $u \in \pi(\{x\} \times \{y\} \times (W - \left\{\overline{-1}, \overline{2}\right\}) \times \left\{m^{-1}\right\})$ and $K \in SB(\pi(\{x\} \times \{y\} \times W \times \left\{m^{-1}\right\}), u)$, then $K \cap \{F(x), F(y)\} \neq \emptyset$.*

Proof. In order to simplify the notation, set $J = \pi(\{x\} \times \{y\} \times W \times \left\{m^{-1}\right\})$. Apply Lemma 2.3 to get $\alpha \in \Lambda(X, u)$ and $s \in [0, 1]$ be such $K = \alpha(s)$ and $\alpha(t) - J \neq \emptyset$ for each $t \in (s, 1]$. First, assume that $K \cap \hat{Y} = \emptyset$. By the continuity of α, there exists $\delta > 0$ such that $\alpha([s, s+\delta]) \subseteq \langle X - \hat{Y} \rangle$. Since $J - \hat{Y}$ is a component of $X - \hat{Y}$ and $\alpha(s) \cap J - \hat{Y} \neq \emptyset$, the set $\alpha(s+\delta)$ is contained in J. A contradiction. Finally, the facts $K \cap \hat{Y} \neq \emptyset$, $K \subseteq Q$ and $Q \cap Y = \{F(x), F(y)\}$ guarantee $K \cap \{F(x), F(y)\} \neq \emptyset$. $\square$

Theorem 3.9. *The set $\mathcal{F}_1(\hat{Y})$ is contained in $\mathcal{NWC}(X)$.*

Proof. Let $u \in \hat{Y}$. Let us prove that $\kappa(E, \{u\}) = X - \{u\}$. First, from the fact that $E \in P$, $P \in C(X)$ and $P \subseteq X - \{u\}$, it follows that $P \subseteq \kappa(E, \{u\})$. Now, let $v \in X - \{u\}$ be such that $v \notin P$. So, $v \in Z$. Take $(x, y, w, z) \in v$ and $c \in C$ satisfying $u = F(c)$. Consider two cases.

Case I. $x, y \in C - f^{-1}(f(c))$.

Define $L = \pi(\{x\} \times \{y\} \times W \times \{z\})$ and $N = \pi(\{y\} \times \{y\} \times W \times \{0\})$ to get a subcontinua of X such that $v \in L$, $E \in N$, $F(y) \in L \cap N$ and $L \cup N \subset X - \{u\}$. Hence, $v \in \kappa(E, \{u\})$.

Case II. $\{x, y\} \cap f^{-1}(f(c)) \neq \emptyset$.

This implies that there exists $e \in W$ such that $(x, y, e, z) \in u = F(c)$. Hence, $e \in \{\overline{-1}, \overline{2}\}$. Assume that $e = \overline{-1}$. From Proposition 3.1, it follows that there exists a subcontinuum J of W such that $w, \overline{2} \in J \subseteq W - \{e\}$. Thus, $L = \pi(\{x\} \times \{y\} \times J \times \{z\})$ and $N = \pi(\{y\} \times \{y\} \times W \times \{0\})$ are subcontinua of X satisfying $v \in L$, $E \in N$, $F(y) \in L \cap N$ and $L \cup N \subseteq X - \{u\}$. So, $v \in \kappa(E, \{u\})$.

Invoke Proposition 2.1 to conclude that $X - \{u\}$ is a continuumwise connected subset of X. Hence, $\{u\} \in \mathcal{NWC}(X)$ □

Theorem 3.10. *If $A \in \mathcal{NB}(\mathcal{F}_1(X))$, then $A \subseteq \hat{Y}$.*

Proof. Combine the fact that the interior of P in X is nonempty, Lemma 2.4, and Proposition 3.6 to conclude that $A \cap P = \emptyset$. Assume that $A \cap Z - \hat{Y} \neq \emptyset$. Choose $(x, y, w, z) \in C \times C \times (W - \{\overline{-1}, \overline{2}\})$ such that $\pi(x, y, w, z) \in A$. Choose $n \in \mathbb{Z}$ fulfilling $w \in W_n$. Combine Lemma 2.4 and Proposition 3.7 to conclude that A must contain the composants of $\pi(\{x\} \times \{y\} \times W_n \times \{z\})$ omitting $\pi(\{x\} \times \{y\} \times \varphi_n(\{\overline{0}, \overline{1}\}) \times \{z\})$. By the compactness of A, $\pi(\{x\} \times \{y\} \times W_n \times \{z\})$ is a subset of A. This implies that $\pi(x, y, \varphi_{n+1}(\overline{0}), z) \in A \cap \pi(\{x\} \times \{y\} \times W_{n+1} \times \{z\})$ and $\pi(x, y, \varphi_{n-1}(\overline{1}), z) \in A \cap \pi(\{x\} \times \{y\} \times W_{n-1} \times \{z\})$. Repeat the argument to conclude that $\pi(\{x\} \times \{y\} \times W_{n+1} \times \{z\})$ and $\pi(\{x\} \times \{y\} \times W_{n-1} \times \{z\})$ are subsets of A. By induction, it can be proved that $\pi(\{x\} \times \{y\} \times W_k \times \{z\}) \subseteq A$ for each $k \in \mathbb{Z}$. Then $\pi(\{x\} \times \{y\} \times W \times \{z\}) \subseteq A$.

Now, if $f(x)$ and $f(y)$ were equal, then E would be an element of $\pi(\{x\} \times \{y\} \times W \times \{z\}) \cap P \subseteq A \cap P$, a contradiction. So, $f(x) \neq f(y)$. From Lemma 2.4 and Proposition 3.8 together, it follows that $\pi(\{x\} \times \{y\} \times W \times \{m^{-1}\}) \subseteq A$ for each $m \in \mathbb{N}$. Then, $E \in \pi(\{x\} \times \{y\} \times W \times \{0\}) \subseteq A$. Hence, $E \in A \cap P$. This contradicts the fact that A and P are disjoint.

Therefore, $A \subseteq \hat{Y}$. □

Theorem 3.11. *The hyperspace $\mathcal{NB}(\mathcal{F}_1(X))$ is a subset of $\mathcal{F}_1(X)$.*

Proof. Let $A \in \mathcal{NB}(\mathcal{F}_1(X))$. Seeking a contradiction, suppose that A has at least two elements. From Lemma 3.5 and Theroem 3.10, it follows that there exist $x, y \in C$ such that $F(x), F(y) \in A$ and $F(x) \neq F(y)$. This implies that $f(x) \neq f(y)$. Apply Lemma 2.4 and Proposition 3.8 to conclude that $\pi(x, y, \overline{0}, 1) \in A$. So, this contradicts the fact that $A - Y$ is empty. □

Theorem 3.12. *The hyperspace $\mathcal{NB}(\mathcal{F}_1(X))$ is homeomorphic to Y.*

Proof. Combine Theorem 3.10 and Theorem 3.11 to obtain that $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(\hat{Y})$. Since $\hat{Y}$ is a topological copy of Y and $\hat{Y}$ is homeomorphic to $\mathcal{F}_1(\hat{Y})$, $\mathcal{NB}(\mathcal{F}_1(X))$ and Y are homeomorphic. □

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Discusión general.

Una gran cantidad de propiedades de un continuo están determinadas por el comportamiento topológico de sus hiperespacio y viceversa. El estudio de estas propiedades ha sido muy amplio. Dentro de esta teoría es muy natural preguntarse si el considerar un subespacio de un hiperespacio cuyos elementos cumplen una característica muy particular se tiene el mismo efecto. Cultivando esta línea de investigación, ha resultado muy interesante estudiar el subespacio que consta de todos los subconjuntos cerrados que no estorban a los singulares. Entre los resultados más importantes sobre este hiperespacio se encuentra la caracterización dentro de la clase de los continuos: la curva cerrada simple es el único continuo cuyo hiperespacio de subconjuntos cerrados que no estorban a sus singulares consta de todos sus singulares. Este resultado implica que el hiperespacio de no estorbadores de los singulares de una curva cerrada simple es un subcontinuo del hiperespacio de cerrado no vacíos homeomorfo a la curva cerrada simple. Surge de forma natura la pregunta sobre la existencia de otros continuos que sean homeomorfos al hiperespacio $NB(F_1(X))$ de un continuo X.

Conclusiones.

En 2019, J. Camargo, F. Capulín, E. Castañeda-Alvarado y D. Maya mostraron una serie de espacios métricos compactos distintos a la curva cerrada simple que cumplen esta condición quedando abierta la pregunta: ¿Para qué continuos Y existe un continuo X tal que $NB(F_1(X))$ es homeomorfo a Y?. El trabajo de investigación realizado está enfocado a dar una respuesta más general: Para cada espacio métrico Y existe un continuo X tal que $NB(F_1(X))$ es homeomorfo a Y. Se utilizaron conceptos dentro de la teoría de continuos que no se habían empleado en la teoría de los no estorbadores como la semi-frontera y esto aporta resultados relevantes, novedosos y originales a esta línea de investigación porque existen pocos artículos sobre el tema.

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Anexos