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Some characterizations of the internal structure of Whitney levels

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Dedicated to Samuel Maya Sánchez

Abstract. Let X be a continuum, and let $C(X)$ denote the hyperspace of all subcontinua of X. It is known that there exist monotone maps μ from $C(X)$ into $[0,\infty)$ such that $\mu(\{x\}) = 0$ for each $x \in X$, and if A is a proper subcontinuum of B, then $\mu(A) < \mu(B)$. The subcontinua $\mu^{-1}(t)$ of $C(X)$ are called Whitney levels of $C(X)$. In this paper, a class of closed subsets of X is employed to characterize the Whitney levels of $C(X)$ possessing one of the following properties: irreducibility, decomposability, being a Wilder continuum, aposyndesis, semiaposyndesis, n-aposyndesis, finite aposyndesis, and connectedness colocal.

Keywords: Aposyndesis, connectedness colocal, decomposability, hyperspace of the subcontinua irreducibility, Whitney level, Wilder continuum.

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Algunas caracterizaciones de la estructura interna de niveles de Whintey

Resumen. Sea X un continuo. Denotamos por $C(X)$ al hiperespacio de todos los subcontinuos de X. Se sabe que existen funciones continuas monótonas μ desde $C(X)$ hacia $[0,\infty)$ tales que $\mu(\{x\}) = 0$, y si A es un subcontinuo propio de B, entonces $\mu(A) < \mu(B)$. Los subcontinuos $\mu^{-1}(t)$ de $C(X)$ son llamados niveles de Whitney. In este artículo, por medio de una clase de subconjuntos cerrados de X se caracterizan los niveles de Whitney que poseen alguna de las siguientes propiedades: ser irreducible, ser

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descomponible, aposindético, semiaposindético, aposindético con respecto a conjuntos finitos, ser colocalmente conexo.

Palabras clave: Aposindesis, conexidad colocal, continuo descomponible, continuo de Wilder, hiperespacio de los subcontinuos, irreducibilidad, niveles de Whitney.

1. Introduction

A *continuum* is a non-degenerate compact connected metric space. In [13] and [14], Hassler Whitney contructed special types of functions on spaces of sets to study families of curves. In [9], J. L. Kelley made significant use of Whitney maps in the study of hyperspaces of continua. Several authors have investigated the relationship between Whitney maps and the topological structure of the hyperspace of all nonempty closed subsets of a continuum, as well as the hyperspace of all subcontinua of a continuum. In particular, the arc structure of these hyperspace is closely related to the Whitney maps.

A *Whitney level* for the hyperspace of all subcontinuum of a continuum is a fiber of a Whitney map. Each Whitney level is a continuum. A practical approach in this study is to examine the topological properties of Whitney levels as subspace of the hyperspace of all subcontinua of a continuum, induced by the topological properties of the ground continuum, and vice versa. The literature contains an extensive list of references addressing this problem (see list of references of [7, Section VIII] for example).

In [6], the author proves a basic theorem that characterizes separating points of the Whitney levels of the hyperspace of all subcontinua of a continuum X as subcontinua of X that separates X in a specific manner. Then, he uses the basic theorem to obtain some information about the Whitney level for the hyperspace of all subcontinua of X if X is indecomposable, arc-like, circle-like, of type A, or hereditarily of type A'. This motivated the characterizations of the points on the edge of Whitney level presented in [3].

In continuum theory, there are properties related to the subcontinua of a continuum. A continuum X is:

- *irreducible* if there exist points $p, q \in X$ such that the unique subcontinuum of X containing p and q is X itself.
- *decomposable* if X is the union of two proper subcontinua.
- *a Wilder continuum* if for any three distinct points x, y, and z of X, there exists a subcontinuum of X containing x and exactly one of y and z,
- *semiaposyndetic* if for every two distinct points x and y of X, there exists a subcontinuum M of X such that the interior of M contains one of x and y, and $X \setminus M$ contains the other point.
- *aposyndetic* if for every two points x and y of X, x is an interior point of a subcontinuum M of X such that M omits y.

- *n-aposyndetic* if for every subset A of X having exactly n elements, and a point $x \in X \setminus A$, x is an interior point of a subcontinuum M of X such that M is disjoint from A.
- *colocally connected* if X has a basis of open subsets with connected complement.

The interrelationship among these notions has been studied in [1], [4], [8], and [11]. The class of aposyndetic continua is a subclass of both the class of colocally connected continua and the class of n-aposyndetic continua. Aposyndesis implies semiaposyndesis. Each semiaposyndetic continuum is a Wilder continuum. Every Wilder continuum is decomposable.

The aim of this paper is to present characterizations of Whitney levels that satisfy one of the aforementioned properties, based on topological properties of the ground continuum. Some characterization theorems are used in the arguments of some examples.

2. Preliminaries and auxiliary results

Let X be a continuum with metric d. The symbol $B(\varepsilon, x)$ denotes the open ball in X of radius $\varepsilon > 0$ centered on $x \in X$. For a subset A of X, set $N(\varepsilon, A) = \bigcup\{B(\varepsilon, a) : a \in A\}$.

Let $C(X)$ be the collection of all subcontinua of a continuum X. Define $H : C(X) \times C(X) \to [0, \infty)$ by $H(A, B) = \text{g.l.b.}\{\varepsilon > 0 : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A)\}$. In [7, Theorem 2.2, p. 11], . The metric space $(C(X), H)$ is *the hyperspace of all subcontinua* of X

Theorem 2.1. *[7, Corollary 14.10, p. 114] Let X be a continuum. The metric space $(C(X), H)$ is an arcwise continuum.*

Theorem 2.2. *Let X be a continuum. If $A, B \in C(X)$ and $\varepsilon > 0$, then $H(A, B) < \varepsilon$ if and only if $A \subseteq N(\varepsilon, B)$ and $B \subseteq N(\varepsilon, A)$.*

Proof. Set $E(A, B) = \{\varepsilon > 0 : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A)\}$. First, assume that $H(A, B) < \varepsilon$. Then ε is not a lower bound of $E(A, B)$. Hence, there exists $\delta \in E(A, B)$ such that $\delta < \varepsilon$. Thus, $A \subseteq N(\delta, B) \subseteq N(\varepsilon, B)$ and $B \subseteq N(\delta, A) \subseteq N(\varepsilon, A)$. Next, assume that $A \subseteq N(\varepsilon, B)$ and $B \subseteq N(\varepsilon, A)$. This guarantees that $\mathcal{C} = \{N(\delta, A) : \delta \in (0, \varepsilon)\}$ is an open cover of the subcontinuum B of X. A finite subcover of $\mathcal{C}$ allows us to choose $\delta_B \in (0, \varepsilon)$ such that $B \subseteq N(\delta_B, A)$. Similarly, there exists $\delta_A \in (0, \varepsilon)$ such that $A \subseteq N(\delta_A, B)$. Set $\delta = \max\{\delta_A, \delta_B\}$ to get an element of $E(A, B) \cap (0, \varepsilon)$. Therefore, $H(A, B) \leq \delta < \varepsilon$. ☑

Throughout this paper, Theorem 2.2 will be used without mentioning it explicitly.

Let X be a continuum. A *Whitney map μ for $C(X)$* is a continuous function $\mu : C(X) \to [0, \infty)$ satisfying $\mu(\{p\}) = 0$ for each $p \in X$, and $\mu(A) < \mu(B)$ whenever A is a proper subset of B. If X is a continuum, then there exist Whitney maps for $C(X)$ (see [7, Theorem 13.4, p. 107]).

Let X be a continuum. A *Whitney level for $C(X)$* is any subset of $C(X)$ that is of the form $\mu^{-1}(t)$, where μ is some Whitney map for $C(X)$ and $t \in [0, \mu(X))$.

Theorem 2.3. *[7, Theorem 19.9, p. 160] If X is a continuum, then each Whitney level for $C(X)$ is a subcontinuum of $C(X)$.*

The next result follows from [7, Theorem 14.6, p. 112]. It will be used frequently throughout the paper without mentioning explicitly.

Lemma 2.4. *Let X be a continuum. If $\mathcal{A}$ is a Whitney level for $C(X)$, then each element of X is contained in an element of $\mathcal{A}$.*

Lemma 2.5. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level for $C(X)$. If $\mathcal{F}$ is a closed subset of $\mathcal{A}$ and $K \in \mathcal{A} \setminus \mathcal{F}$, then there exists $\delta > 0$ such that each element of $\mathcal{F}$ intersects $X \setminus N(\eta, K)$ for each $\eta \in (0, \delta]$.*

Proof. Seeking a contradiction, suppose to the contrary. So, for each $n \in \mathbb{N}$, there exist $\eta_n \in (0, \frac{1}{n}]$ and $F_n \in \mathcal{F}$ such that $F_n \subseteq N(\eta_n, K)$. The compactness of $\mathcal{A}$ allows us to assume that the sequence $(F_n)_{n\in\mathbb{N}}$ converges. Set $F = \lim F_n$. Then $F \in \mathcal{F}$. Let us prove that $F \subseteq K$.

Let $x \in F$ and let $\varepsilon > 0$. Then there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < \frac{\varepsilon}{2}$ and $H(F_m, F) < \frac{\varepsilon}{2}$. This implies that $F \subseteq N(\frac{\varepsilon}{2}, F_m)$ and $F_m \subseteq N(\frac{\varepsilon}{2}, K)$. Hence, there exists $z \in F_m \cap B(\frac{\varepsilon}{2}, x)$ and there exists $y \in B(\frac{\varepsilon}{2}, z) \cap K$. Thus, $y \in B(\varepsilon, x) \cap K$. In conclusion, x is a point of the closure of K. Therefore, $x \in K$. This proves that F is a subset of K.

Finally, the unique element of $\mathcal{A}$ contained in K is K. This proves $K \in \mathcal{F}$. A contradiction. ☑

Lemma 2.6. *Let E be a nonempty closed subset of a continuum X and let $\mathcal{F}$ be a closed subset of a Whitney level $\mathcal{A}$ for $C(X)$. If each element of $\mathcal{F}$ intersects E, then the union of all components of E containing a point of an element of $\mathcal{F}$ is a closed subset of X.*

Proof. Let G be the union of all components of E containing a point of an element of $\mathcal{F}$ and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of elements of G. For each $n \in \mathbb{N}$, let C_n be the component of E containing x_n and let $F_n \in \mathcal{F}$ be such that $C_n \cap F_n \neq \emptyset$. The compactness of $C(X)$ allows us to assume that $(C_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ converge. If $x = \lim x_n$, $C = \lim C_n$ and $F = \lim F_n$, then C is a subcontinuum of X such that $x \in C \subseteq E$ and $C \cap F \neq \emptyset$. Hence, the component of E containing x intersects an element of $\mathcal{L}$. This proves that $x \in G$. So, G is a closed subset of X. ☑

Let X be a continuum, and let $\mathcal{A}$ be a Whitney level for $C(X)$. A nonempty closed subset G of X is an $\mathcal{A}$-*subcontinuum* of X if for each pair of nonempty disjoint closed subsets R and S of X such that $G = R \cup S$, there exists an element $A \in \mathcal{A}$ such that $A \cap R \neq \emptyset$ and $A \cap S \neq \emptyset$. This notion is introduced in [3].

Lemma 2.7. *Let G be a nonemtpy closed subset of a continuum X and let $\mathcal{L}$ be a subcontinuum of a Whitney level $\mathcal{A}$ for $C(X)$. If each element of $\mathcal{L}$ intersects G and each component of G contains a point of an element of $\mathcal{L}$, then G is an $\mathcal{A}$-subcontinuum of X.*

Proof. In light of Lemma 2.6, G is a closed subset of X. Now, assume that R and S are nonempty disjoint closed subsets of X such that $G = R \cup S$. So, each $T \in \mathcal{L}$ fulfills T intersects G. So, either $T \in \Lambda(S)$ or $T \in \Lambda(R)$ for each $T \in \mathcal{L}$. This proves that $\mathcal{L} \subseteq \Lambda(S) \cup \Lambda(R)$. Next, each component D of G satisfies either $D \subseteq S$ or $D \subseteq R$. If D is a component of G containing a point of S, then D intersects an element L of $\mathcal{L}$ and hence, $L \in \mathcal{L} \cap \Lambda(S)$. Similarly, $\mathcal{L} \cap \Lambda(R) \neq \emptyset$.

This proves that the subcontinuum $\mathcal{L}$ of $\mathcal{A}$ is contained in the union of the closed subsets of $\Lambda(S)$ and $\Lambda(R)$ of $\mathcal{A}$ and $\mathcal{L}$ intersects both $\Lambda(S)$ and $\Lambda(R)$. Hence, $\Lambda(S)$ is not disjoint from $\Lambda(R)$. Thus, $\mathcal{A}$ has an element that intersects both R and S. Therefore, G is an $\mathcal{A}$-subcontinuum of X. ☑

Let X be a continuum and let $\mathcal{A}$ be a Whitney level for $C(X)$. For a subset C of X, let

$$\Lambda_{\mathcal{A}}(C) = \{A \in \mathcal{A} : A \cap C \neq \emptyset\}.$$

If there is no risk of confusion, the subindex will be removed for short. This means that the symbol $\Lambda(C)$ will be used instead of $\Lambda_{\mathcal{A}}(C)$. If C is a nonempty closed subset of X, then $\Lambda(C)$ is a nonempty closed subset of $\mathcal{A}$.

Lemma 2.8. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level for $C(X)$. If M is a subcontinuum of X contained in an element of $\mathcal{A}$, then $\Lambda(M)$ is an arcwise subcontinuum of $\mathcal{A}$.*

Proof. Let $K \in \mathcal{A}$ be such that $M \subseteq K$ and let $L \in \Lambda(M)$. Then $L \cap K \neq \emptyset$. Choose $p \in L \cap K$. By [12, Theorem 3.6, p. 575], there exists a map $\varphi : [0,1] \to \mathcal{A}$ such that $\varphi(0) = K$, $\varphi(1) = L$, and $p \in \varphi(t) \subseteq K \cup L$ for each $t \in [0,1]$. Hence, each element of $\Lambda(M)$ and K are elements of an arc contained in $\Lambda(M)$. ☑

Lemma 2.9. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level for $C(X)$. If C is an $\mathcal{A}$-subcontinuum of X, then $\Lambda(C)$ is a subcontinuum of $\mathcal{A}$.*

Proof. It only remains to prove that $\Lambda(C)$ is connected. Let Σ and Ψ be nonempty closed subsets of $\mathcal{A}$ such that $\Lambda(C) = \Sigma \cup \Psi$. Set $S = \{x \in C : x \text{ is a point of a member of } \Sigma\}$ and $R = \{x \in C : x \text{ is a point of a member of } \Psi\}$ to get nonempty closed subsets of X contained in C. Now, if $x \in C$ and $T \in \mathcal{A}$ contains x, then either $T \in \Sigma$ or $T \in \Psi$. Hence, C is the union of S and R. Since C is an $\mathcal{A}$-subcontinuum of X, there exists an element F of $\mathcal{A}$ such that $F \cap S \neq \emptyset$ and $F \cap R \neq \emptyset$. Then either $F \in \Sigma$ or $F \in \Psi$. Assume that $F \in \Sigma$. Choose $w \in F \cap R$ and $E \in \Psi$ containing w. Lemma 2.8 ensures that there exists a map such that $\varphi(0) = F$, $\varphi(1) = E$, and $w \in \varphi(t)$ for each $t \in [0,1]$. Then $\varphi([0,1])$ is a subset of $\Lambda(C)$. Hence, the connected space $[0,1]$ is the union of the nonempty closed subsets $\varphi^{-1}(\Sigma)$ and $\varphi^{-1}(\Psi)$. So, there exists $s \in [0,1]$ such that $\varphi(s) \in \Sigma \cap \Psi$. In conclusion, $\Lambda(C)$ is connected. ☑

3. Main results

Notice that a continuum X is colocally connected if and only if for each point $x \in X$ and each closed subset F of X omitting x, there exists a subcontinuum L of X such that L omits x and contains F.

Theorem 3.1. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level for $C(X)$. Then $\mathcal{A}$ is a colocally connected continuum if and only if for each $A \in \mathcal{A}$ and each closed subset F of X disjoint from A, there exists an $\mathcal{A}$-subcontinuum G of X disjoint from A and containing F.*

Proof. Assume that $\mathcal{A}$ is a colocally connected continuum. Let $A \in \mathcal{A}$ and let F be a nonempty closed subset of X disjoint from A. Then $\Lambda(F)$ is a closed subset of $\mathcal{A}$ omitting A. Hence, there exists a subcontinuum Δ of $\mathcal{A}$ omitting A and containing $\Lambda(F)$. Lemma 2.5 allows us to take $\eta > 0$ such that each element of Δ intersects $X \setminus N(\eta, A)$ and F is disjoint from $N(\eta, A)$. Let G be the union of components of $X \setminus N(\eta, A)$ containing a point of an element of Δ. Notice that G contains F. Apply Lemma 2.6 to infer that G is a closed subset of X. This and Lemma 2.7 together guarantee that G is an $\mathcal{A}$-subcontinuum of X. The proof of the first part is complete.

In order to prove the second part, let $A \in \mathcal{A}$ and let $\mathcal{F}$ be a closed subset of $\mathcal{A}$ omitting A. Use Lemma 2.5 to choose $\eta > 0$ such that each element of $\mathcal{F}$ intersects $X \setminus N(\eta, A)$. Then $X \setminus N(\eta, A)$ is a closed subset of X disjoint from A. Thus, there exists an $\mathcal{A}$-subcontinuum G of X disjoint A and containing $X \setminus N(\eta, A)$. Use Lemma 2.9 to deduce that $\Lambda(G)$ is a subcontinuum of $\mathcal{A}$. Finally, since each element of $\mathcal{F}$ intersects $X \setminus N(\eta, A)$, $\mathcal{F}$ is a subset of $\Lambda(G)$. Therefore, $\mathcal{A}$ is colocally connected. ☑

The continuum X exhibited in [7, Figure 36, p. 241] is nonaposyndetic, whereas each Whitney level for $C(X)$ is aposyndetic. This fact is strengthened by proving that each Whitney level for $C(X)$ is colocally connected using Theorem 3.1. Recall that each colocally connected continuum is aposyndetic.

Example 3.2 (E. Matsuhashi). *Let $C_0 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, and for each $n \in \mathbb{N}$, let $C_n = \{(x, y) \in \mathbb{R}^2 : (x - \frac{1}{n})^2 + y^2 = (1 + \frac{1}{n})^2\}$. Set $X = C_0 \cup \bigcup\{C_n : n \in \mathbb{N}\}$. Notice that X is a nonaposyndetic continuum.*

Let $\mathcal{A}$ be a Whitney level, let $A \in \mathcal{A}$ and let F be a closed subset of X disjoint from A. If $(-1, 0) \notin A$, then there exists $\delta > 0$ such that $X \setminus N(\delta, A)$ is a subcontinuum of X containing $F \cup \{(-1, 0)\}$. This proves that under the assumption $(-1, 0) \notin A$, there exists an $\mathcal{A}$-subcontinuum of X disjoint from A containing F.

Next, assume that $(-1, 0) \in A$. Choose $\delta > 0$ such that F is disjoint from $N(\delta, A)$ and each component of $X \setminus N(\delta, A)$ is an arc. Let us prove that $G = X \setminus N(\delta, A)$ is an $\mathcal{A}$-subcontinuum of X.

Let R and S be nonempty disjoint closed subsets of X such that $G = R \cup S$. Choose $n, m \in \mathbb{N} \cup \{0\}$ such that $R \cap C_n \neq \emptyset$ and $S \cap C_m \neq \emptyset$. Then there exists an arc I in $C_n \cup C_m$ contained in an element K of $\mathcal{A}$ starting in R and ending in S. So, K is an element of $\mathcal{A}$ that intersects both R and S. Thus, G is an $\mathcal{A}$-subcontinuum of X.

The required properties in Theorem 3.1 hold. Hence, $\mathcal{A}$ is a colocally connected continuum.

A continuum X is *freely decomposable* if for each pair of distinct points x and y of X, there exist subcontinua A and B such that $X = A \cup B$, $x \in A \setminus B$ and $y \in B \setminus A$. Free decomposability and aposyndesis are equivalent (see [10, Theorem 1.4.28]).

Theorem 3.3. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level for $C(X)$. Then $\mathcal{A}$ is freely decomposable if and only if for each pair of distinct points A and B of $\mathcal{A}$, there exist $\mathcal{A}$-subcontinua G and J of X such that each element of $\mathcal{A}$ disjoint from G intersects J, each element of $\mathcal{A}$ disjoint from J intersects G, G is disjoint from B, and J is disjoint from A.*

Proof. Assume that $\mathcal{A}$ is freely decomposable. Let A and B be distinct elements of $\mathcal{A}$. Then there exist subcontinua Δ and Σ of $\mathcal{A}$ such that $\mathcal{A} = \Delta \cup \Sigma$, $A \in \Delta \subseteq \mathcal{A} \setminus \{B\}$ and $B \in \Sigma \subseteq \mathcal{A} \setminus \{A\}$. Use Lemma 2.5 to obtain $\eta > 0$ such that each element of Δ intersects $X \setminus N(\eta, B)$ and each element of Σ intersects $X \setminus N(\eta, A)$. Let G be the union of all components of $X \setminus N(\eta, B)$ containing a point of an element of Δ and let J be the union of all components of $X \setminus N(\eta, A)$ containing a point of an element of Σ. Then G and J are closed subsets of X by Lemma 2.6. Notice that G is disjoint from B and J is disjoint from A. Apply Lemma 2.7 to infer that G and J are $\mathcal{A}$-subcontinua of X. Observe that if $K \in \mathcal{A}$ is disjoint from J, then K cannot be an element of Σ and so K must be an element of Δ. This proves that each element of $\mathcal{A}$ disjoint from J intersects G. Thus, G and J satisfy the required properties.

Now, let A and B be distinct elements of $\mathcal{A}$. Assume that there exist $\mathcal{A}$-subcontinua G and J of X satisfying each element of $\mathcal{A}$ disjoint from G intersects J, each element of $\mathcal{A}$ disjoint from J intersects G, G is disjoint from B, and J is disjoint from A. Invoke Lemma 2.9 to infer that $\Lambda(G)$ and $\Lambda(J)$ are subcontinua of X. Notice that $\mathcal{A} = \Lambda(G) \cup \Lambda(J)$, $A \in \Lambda(G) \subseteq \mathcal{A} \setminus \{B\}$, and $B \in \Lambda(J) \subseteq \mathcal{A} \setminus \{A\}$. In conclusion, $\mathcal{A}$ is freely decomposable. ☑

The next result is an immediate consequence of the last theorem and the equivalence between aposyndesis and free decomposability.

Corollary 3.4. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level for $C(X)$. Then $\mathcal{A}$ is aposyndetic if and only if for each pair of distinct point A and B of $\mathcal{A}$, there exist $\mathcal{A}$-subcontinua G and J of X such that each element of $\mathcal{A}$ disjoint from G intersects J, each element of $\mathcal{A}$ disjoint from J intersects G, G is disjoint from B, and J is disjoint from A.*

Theorem 3.5. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level of $C(X)$. Then $\mathcal{A}$ is semiaposyndetic if and only if for each distinct elements A and B of $\mathcal{A}$, there exist a $\mathcal{A}$-subcontinuum G of X and a closed subset F of X such that each element of $\mathcal{A}$ disjoint from F intersects G, and each one of F and G intersects only one of A and B.*

Proof. First, assume that $\mathcal{A}$ is semiaposyndetic and let A and B be distinct elements of $\mathcal{A}$. Assume that there exists a subcontinuum $\mathcal{W}$ of $\mathcal{A}$ such that A is an interior point of of $\mathcal{W}$, and $\mathcal{A} \setminus \mathcal{W}$ contains B. Use Lemma 2.5 to get $\eta > 0$ satisfying each element of $\mathcal{W}$ intersects $X \setminus N(\eta, B)$ and each element of the closure of $\mathcal{A} \setminus \mathcal{W}$ intersects $X \setminus N(\eta, A)$. Let G be the union of all components of $X \setminus N(\eta, B)$ containing a point of an element of $\mathcal{W}$. Then G intersects A and G is disjoint from B. Lemma 2.6 guarantees that G is a closed subset of X. Apply Lemma 2.7 to conclude that G is an $\mathcal{A}$-subcontinuum of X. Set $F = X \setminus N(\eta, A)$ to get a nonempty closed subset of X that intersects B and is disjoint from A. If $K \in \mathcal{A}$ is disjoint from F, then K cannot be an element of the closure of $\mathcal{A} \setminus \mathcal{W}$, in other words, K is an interior point of W and so K intersects G. Thus, G and F satisfy the required properties.

Now, let A and B be distinct elements of $\mathcal{A}$. Assume that there exist an $\mathcal{A}$-subcontinuum G of X and a closed subset F of X such that each element of $\mathcal{A}$ disjoint from F intersects G, each one of F and G intersects only one of A and B. Suppose that G intersects A and G is disjoint from B. Then F intersects B and F is disjoint from A. Lemma 2.9 ensures that $\Lambda(G)$ is a subcontinuum of $\mathcal{A}$. Notice that $B \in \mathcal{A} \setminus \Lambda(G)$ and $A \in \mathcal{A} \setminus \Lambda(F)$. Observe that if $K \in \mathcal{A} \setminus \Lambda(F)$, then $K \in \mathcal{G}$. This proves that the open subset $\mathcal{A} \setminus \Lambda(F)$ of $\mathcal{A}$ contains A and is a subset of the subcontinuum $\Lambda(G)$ of $\mathcal{A}$ contained in $\mathcal{A} \setminus \{B\}$. Therefore, $\mathcal{A}$ is semiaposyndetic. ☑

Similar arguments given in the proof of the last result can be used to prove the next theorem, and they will be repeated for completeness.

Theorem 3.6. *Let X be a continuum, let $\mathcal{A}$ be a Whitney level of $C(X)$, and let $n \in \mathbb{N}$. Then $\mathcal{A}$ is n-aposyndetic if and only if for each subset $\mathcal{F}$ of $\mathcal{A}$ having at most n elements and for each $A \in \mathcal{A} \setminus \mathcal{F}$, there exist an $\mathcal{A}$-subcontinua G of X and a closed subset F of X such that each element of $\mathcal{A}$ disjoint from F intersects G, each element of $\mathcal{F}$ is disjoint from G, and F is disjoint from A.*

Proof. Assume that $\mathcal{A}$ is n-aposyndetic. Let $\mathcal{F}$ be a subset of X having at most n-elements and let $A \in \mathcal{A} \setminus \mathcal{F}$. Then there exists a subcontinuum of $\mathcal{W}$ of $\mathcal{A}$ such that A is an interior point of $\mathcal{W}$ and $\mathcal{W}$ is disjoint from $\mathcal{F}$. Lemma 2.5 guarantees that there exists $\eta > 0$ such that each element of the closure of $\mathcal{A} \setminus \mathcal{W}$ intersects $X \setminus N(\eta, A)$ and each element of $\mathcal{W}$ intersects to $X \setminus \bigcup\{N(\eta, B) : B \in \mathcal{F}\}$. Set $F = X \setminus N(\eta, A)$ to get a closed subset of X disjoint from A and let G be the union of all components of $X \setminus \bigcup\{N(\eta, B) : B \in \mathcal{F}\}$ having at least one point of an element of $\mathcal{W}$. By Lemma 2.6, G is a closed subset of X. Use Lemma 2.7 to infer that G is an $\mathcal{A}$-subcontinuum of X. Observe that each element of $\mathcal{F}$ is disjoint from G. Now, if $D \in \mathcal{A}$ is disjoint from F, then D is not an element of the closure of $\mathcal{A} \setminus \mathcal{W}$, in other words, D must be an element of $\mathcal{W}$, and so, D intersects G. Thus, F and G satisfy the required properties. This ends the proof of the first part.

Now, in order to prove that $\mathcal{A}$ is n-aposyndetic, let $\mathcal{F}$ be a subset of X having at most n-elements, and let $A \in \mathcal{A} \setminus \mathcal{F}$. Choose an $\mathcal{A}$-subcontinuum G of X and a closed subset F of X such that each element of $\mathcal{A}$ disjoint from F intersects G, each element of $\mathcal{F}$ is disjoint from G, and F is disjoint from A. Apply Theorem 2.9 to prove that $\Lambda(G)$ is a subcontinuum of $\mathcal{A}$. Notice that $\Lambda(G)$ is disjoint from $\mathcal{F}$ and $A \in \mathcal{A} \setminus \Lambda(F)$. Each element of $\mathcal{A} \setminus \Lambda(F)$ intersects G. In other words, $\mathcal{A} \setminus \Lambda(F)$ is an open subset of $\mathcal{A}$ that contains A and is contained in $\Lambda(G)$. Therefore, A is an interior point of the subcontinuum $\Lambda(G)$ of $\mathcal{A}$ and $\Lambda(G)$ is disjoint from $\mathcal{F}$. Thus, $\mathcal{A}$ is n-aposyndetic. ☑

A continuum X is *finite set aposyndetic* provided X is n-aposyndetic for each $n \in \mathbb{N}$. The next result is an immediate consequence of the last theorem.

Corollary 3.7. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level of $C(X)$. Then $\mathcal{A}$ is finite set aposyndetic if and only if for each finite subset $\mathcal{F}$ of $\mathcal{A}$ and for each $A \in \mathcal{A} \setminus \mathcal{F}$, there exist an $\mathcal{A}$-subcontinua G of X and a closed subset F of X such that each element* *of $\mathcal{A}$ disjoint from F intersects G, each element of $\mathcal{F}$ is disjoint from G, and F is disjoint from F.*

Theorem 3.8. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level of $C(X)$. Then $\mathcal{A}$ is a Wilder continuum if and only if for each different elements T, P and Q of $\mathcal{A}$, there exists an $\mathcal{A}$-subcontinuum G of X satisfying T intersects G and G intersects exactly one of P and Q.*

Proof. Assume that $\mathcal{A}$ is a Wilder continuum. Take different elements T, P and Q of $\mathcal{A}$. Suppose that there exists a subcontinuum $\mathcal{L}$ of $\mathcal{A}$ containing T and satisfying $P \in \mathcal{L} \subseteq \mathcal{A} \setminus \{Q\}$. Lemma 2.5 allows to find $\eta > 0$ such that each element of $\mathcal{L}$ intersects $X \setminus N(\eta, Q)$. Let G be the union of all components of $X \setminus N(\eta, Q)$ containing a point of an element of $\mathcal{L}$. Notice that G is disjoint from Q and G intersects both P and T. Lemma 2.6 and Lemma 2.7 together prove that G is an $\mathcal{A}$-subcontinuum of X. Therefore, G satisfies the required properties.

In order to prove the second part, let P, Q and T be different elements of $\mathcal{A}$. The assumption guarantees the existence of an $\mathcal{A}$-subcontinuum G of X such that T intersects G and G intersects exactly one of P and Q. Use Lemma 2.9 to get that $\Lambda(G)$ is a subcontinuum of $\mathcal{A}$. Notice that $T \in \Lambda(G)$ and $\Lambda(G)$ contains exactly one of P and Q. This proves that $\mathcal{A}$ is Wilder. ☑

Theorem 3.9. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level of $C(X)$. Then $\mathcal{A}$ is decomposable if and only if there exist $\mathcal{A}$-subcontinua G and J of X such that each element of $\mathcal{A}$ disjoint from G interesects J, G is disjoint from an element of $\mathcal{A}$, and J is disjoint from an element of $\mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is the union of the proper subcontinua $\mathcal{K}$ and $\mathcal{L}$. Choose $A \in \mathcal{A} \setminus \mathcal{K}$ and $B \in \mathcal{A} \setminus \mathcal{L}$. Lemma 2.5 allows us to choose $\eta > 0$ such that each element of $\mathcal{K}$ intersects $X \setminus N(\eta, A)$ and each element of $\mathcal{L}$ intersects $X \setminus N(\eta, B)$. Let G be the union of all components of $X \setminus N(\eta, A)$ containing a point of a member of $\mathcal{K}$ and let J be the union of all components of $X \setminus N(\eta, B)$ containing a point of an element of $\mathcal{L}$. Use Lemma 2.6 to conclude that G and J are closed subsets of X. Invoke Lemma 2.7 to infer that G and J are $\mathcal{A}$-subcontinua of X. Notice that G is disjoint from A and J is disjoint from B. Finally, if $D \in \mathcal{A}$ is disjoint from G, then D cannot be an element of $\mathcal{K}$ and so $D \in \mathcal{L}$. This implies that each element of $\mathcal{A}$ disjoint from G intersects J. This proves the first part.

Now, assume that G and J are $\mathcal{A}$-subcontinua of X such that each element of $\mathcal{A}$ disjoint from G intersects J, G is disjoint from an element of $\mathcal{A}$, and J is disjoint from an element of $\mathcal{A}$. This and Lemma 2.9 together guarantee that $\Lambda(G)$ and $\Lambda(J)$ are proper subcontinua of $\mathcal{A}$ and $\mathcal{A}$ is the union of $\Lambda(G)$ and $\Lambda(J)$. Therefore, $\mathcal{A}$ is decomposable. ☑

Example 3.10. *There exists an indecomposable continuum X such that each Whitney level for $C(X)$ is decomposable.*

Let X be the continuum described in [5, Example 2, p. 360]; i. e., X is a slight modification of the well known buckethandle continuum, but it contains a simple triod with vertex p and end points the usual end point of the buckethandle continuum r, q and s pictured in Figure 1 (compare with [2, Fig. 1, p. 7]).

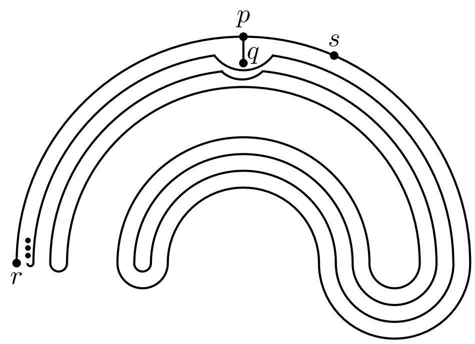

Figure 1. Continuum X

Now, let $\mathcal{A}$ be a Whitney level for $C(X)$. Choose $A, B \in \mathcal{A}$ satisfying A is triod omitting q, and B is an arc having end point s and omitting p. Choose $\eta > 0$ such that $N(\eta, A) \subseteq X \setminus \{q\}$ and $N(\eta, B) \subseteq X \setminus \{p\}$. Let G be the component of $X \setminus N(\eta, B)$ containing r and let $J = X \setminus N(\eta, A)$. Notice that G is a $\mathcal{A}$-subcontinuum of X disjoint from B, J is a nonempty closed subset of X disjoint from A, and each element of $\mathcal{A}$ disjoint from G intersects J. If R and S are nonempty closed subsets of X such that $J = R \cup S$, then there exists an arc I contained in X such that I intersects both R and S. So, J is an $\mathcal{A}$-subcontinuum of X. Apply Theorem 3.9 to conclude that $\mathcal{A}$ is decomposable.

The paper conclude with the following theorem.

Theorem 3.11. *Let X be a continuum and let $\mathcal{A}$ be a Whitney level of $C(X)$. Then $\mathcal{A}$ is irreducible if and only if there exist elements P and Q of $\mathcal{A}$ such that each $\mathcal{A}$-subcontinuum G of X containing a point of P and a point of Q intersects each element of $\mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is irreducible. Then there exist elements P and Q of $\mathcal{A}$ such that the unique subcontinuum of $\mathcal{A}$ containing P and Q is $\mathcal{A}$. Let G be an $\mathcal{A}$-subcontinuum of G such that G contains a point of P and a point of Q. Lemma 2.9 ensures that $\Lambda(G)$ is a subcontinuum of $\mathcal{A}$. Notice that P and Q are elements of $\Lambda(G)$. So, $\mathcal{A}$ and $\Lambda(G)$ coincide. Hence, each element of $\mathcal{A}$ intersects G.

Now, assume that P and Q are elements of $\mathcal{A}$ such that each $\mathcal{A}$-subcontinuum G of X containing a point of P and a point of Q intersects each element of $\mathcal{A}$. Let $\mathcal{K}$ be a subcontinuum of $\mathcal{A}$ containing P and Q. Assume that there exists $A \in \mathcal{A} \setminus \mathcal{K}$. Use Lemma 2.5 to choose $\eta > 0$ such that each element of $\mathcal{K}$ intersects $X \setminus N(\eta, A)$. So, $X \setminus N(\eta, A)$ intersects both P and Q. Let G be the union of all components of $X \setminus N(\eta, A)$ having a point of an element of $\mathcal{K}$. Notice that A is disjoint from G and G intersects both P and Q. Apply Lemma 2.6 to infer that G is a closed subset of X. Lemma 2.7 ensures that G is an $\mathcal{A}$-subcontinuum of X. This is a contradiction. Therefore, $\mathcal{K}$ and $\mathcal{A}$ coincide. In conclusion, $\mathcal{A}$ is irreducible. ☑

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